Correction model

Mathematics B

April 9th, 2020
10:00-12:30

1 General

1. The number of points awarded for a question can only be $0, 1, 2, \ldots, n$ where n equals the maximum number of points for a question.
2. For each calculation error, notation error or typo, 1 point will be deducted from the maximum number of points that can be achieved for the question
3. If part of the solution is shown between brackets in the correction model, this part need not be included in the candidate's solution.
4. An error may only be charged once in the solution of the question, unless the question simplifies considerably as a result of the error.
5. The same error in answering different questions must be counted again and again, unless stated otherwise in the correction model.
6. If a question is solved with rounded intermediate answers, and this leads to a different final answer than when calculated with unrounded intermediate answers, then 1 point will be deducted from this question, unless stated otherwise in the assessment model.

2 Correction model

5p 1. $f(x) = 3x^2 - x^3$ gives $f'(x) = 6x - 3x^2$ and $f''(x) = 6 - 6x$. 1
$f''(x) = 0 \implies x = 1$ 1
$f(1) = 2$ 1
Define $l\colon y = ax + b$.
$a = f'(1) = 3$ 1
$l\colon y = 3x + b$ passing through $(1, 2)$ gives $b = -1$.
Hence, $l\colon y = 3x - 1$. 1

5p 2. $3x + 5 = 0 \implies x = -\frac{5}{3}$ (which is not a zero of the numerator),
so the vertical asymptote of f is $x = -\frac{5}{3}$. 1
$2x - 1 \geq 0 \implies x \geq \frac{1}{2}$, so $|2x - 1| = \begin{cases} 2x - 1 & \text{for } x \geq \frac{1}{2} \\ 1 - 2x & \text{for } x < \frac{1}{2} \end{cases}$ 1
$\lim\limits_{x\to\infty}\left(\frac{|2x-1|}{3x+5} + 1\right) = \lim\limits_{x\to\infty}\left(\frac{2x-1}{3x+5} + 1\right) = \lim\limits_{x\to\infty}\left(\frac{2-\frac{1}{x}}{3+\frac{5}{x}} + 1\right) = \frac{2-0}{3+0} + 1 = \frac{5}{3}$ 1
$\lim\limits_{x\to-\infty}\left(\frac{|2x-1|}{3x+5} + 1\right) = \lim\limits_{x\to-\infty}\left(\frac{1-2x}{3x+5} + 1\right) = \lim\limits_{x\to-\infty}\left(\frac{\frac{1}{x}-2}{3+\frac{5}{x}} + 1\right) = \frac{0-2}{3+0} + 1 = \frac{1}{3}$ 1
Hence, the horizontal asymptotes of f are the lines $y = \frac{1}{3}$ and $y = \frac{5}{3}$. 1

4p 3. $x(t) = 0 \implies t^2 = 1 \implies t = 1 \vee t = -1$ 1
$y(1) = 3 > 0$, so P crosses the positive y-axis at $t = 1$. 1
$\vec{v}(t) = \begin{pmatrix} 2t \\ 2t + 2 \end{pmatrix}$ 1
$\vec{v}(1) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$
The orbital velocity is $|\vec{v}(1)| = \sqrt{2^2 + 4^2} = \sqrt{20}$ (or $2\sqrt{5}$). 1

4p 4. a. $O(V) = \int_0^1 \sqrt{1-x}\,\mathrm{d}x$ 1
$= \left[-\frac{2}{3}(1-x)\sqrt{1-x}\right]_0^1$ 2
$= \frac{2}{3}$ 1

Remark
If the candidate does not apply the chain rule for antiderivatives, then at most one point may be awarded for the antiderivative.

4p b. $y = \sqrt{1-x} \implies y^2 = 1-x \implies x = 1-y^2$ 1

$I(L) = \pi \displaystyle\int_0^{f(0)} x^2 \,\mathrm{d}y$

$= \pi \displaystyle\int_0^{1} \left(1-2y^2+y^4\right) \mathrm{d}y$ 1

$= \pi \left[y - \tfrac{2}{3}y^2 + \tfrac{1}{5}y^5\right]_0^1$ 1

$= \pi \left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right)$

$= \tfrac{8}{15}\pi$ 1

5p c. $f(x) = g(x) \implies \sqrt{1-x} = 1-x$

$1-x = (1-x)^2$

$1-x = 1-2x+x^2$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0 \ \vee \ x = 1$ 1

(valid valid)

$I(M) = \pi \displaystyle\int_0^{1} \left((\sqrt{1-x})^2 - (1-x)^2\right) \mathrm{d}x$ 1

$= \pi \displaystyle\int_0^{1} \left(1-x-1+2x-x^2\right) \mathrm{d}x$

$= \pi \displaystyle\int_0^{1} \left(x-x^2\right) \mathrm{d}x$ 1

$= \pi \left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1$ 1

$= \pi \left(\tfrac{1}{2} - \tfrac{1}{3}\right)$

$= \tfrac{1}{6}\pi$ 1

6p 5. $f'(x) = 6\cos(x) + 2\sin(2x)$ 2

$f'(x) = 0 \implies 6\cos(x) + 2\sin(2x) = 0$

$6\cos(x) + 4\sin(x)\cos(x) = 0$ 1

$2\cos(x)(3 + 2\sin(x)) = 0$

$\cos(x) = 0 \;\vee\; \sin(x) = -\frac{3}{2}$ 1

$\cos(x) = 0$ implies $x = \frac{1}{2}\pi + k \cdot \pi$ 1
$\sin(x) = -\frac{3}{2}$ has no solutions. 1

Remark
If the candidate does not apply the chain rule for derivatives, then at most one point may be awarded for the derivative.

4p 6. a. Define $k\colon y = ax + b$.
The centre of c is $M(14, 8)$ and $\text{slope}_{AM} = \dfrac{8-0}{14-8} = \frac{4}{3}$. 2
$k \perp AM$ implies $\text{slope}_{AM} \cdot \text{slope}_k = -1$, so $a = -\frac{3}{4}$. 1
$k\colon y = -\frac{3}{4}x + b$ passing through $A(8, 0)$ gives $b = 6$.
Hence, $k\colon y = -\frac{3}{4}x + 6$. 1

Remark
If the candidate only considers the centre of c, then no points may be awarded for this question.

3p b. Substituting $y = 0$ into the circle equation gives

$(x - 14)^2 + (-8)^2 = 100$

$(x - 14)^2 = 36$

$x - 14 = -6 \;\vee\; x - 14 = 6$

$x = 8 \;\vee\; x = 20$ 1

So, $AB = 12$. 1
It follows that $AB = 12 = \frac{3}{2} \cdot 8 = \frac{3}{2} \cdot OA$. 1

3p c. $\vec{r}_{OM} = \begin{pmatrix} 14 \\ 8 \end{pmatrix}$ 1

$l \perp OM \implies \vec{r}_l = \vec{n}_{OM} = \begin{pmatrix} 8 \\ -14 \end{pmatrix}$ 1

The midpoint of OM is $(7, 4)$, so $\vec{s}_l = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$.

A vector representation is $l\colon \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ -14 \end{pmatrix}$. 1

4p d. Substituting $y = 2x$ into $(x - 14)^2 + (y - 8)^2 = 100$ gives

$(x - 14)^2 + (2x - 8)^2 = 100$ 1

$x^2 - 28x + 196 + 4x^2 - 32x + 64 = 100$

$5x^2 - 60x + 160 = 0$ 1

$x^2 - 12x + 32 = 0$

$(x - 4)(x - 8) = 0$

$x = 4 \ \vee \ x = 8$ 1

Since $y = 2x$, we obtain $C(4, 8)$ and $D(8, 16)$. 1

6p 7. The length of AB is given by $L(p) = g(p) - f(p) = 4 - e^{1-p} - e^{p-1}$. 1

$L'(p) = e^{1-p} - e^{p-1}$ 1

$L'(p) = 0 \implies e^{1-p} = e^{p-1}$ 1

$1 - p = p - 1$

$2p = 2$

$p = 1$ 1

$L''(p) = -e^{1-p} - e^{p-1}$

$L''(1) = -2 < 0$, so the length of AB is maximal for $p = 1$. 1

The maximum length of AB equals $L(1) = 4 - e^0 - e^0 = 2$. 1

8p 8. $f_p'(x) = \dfrac{-p\cos(x)}{\sin^2(x)}$ 1

$$\begin{aligned} f_p'(x) = 0 \implies & \frac{-p\cos(x)}{\sin^2(x)} = 0 \\ & -p\cos(x) = 0 \\ & \cos(x) = 0 \\ & x = \tfrac{1}{2}\pi + k\cdot\pi \end{aligned}$$ 1

This gives $x_A = \frac{1}{2}\pi$, $x_B = \frac{3}{2}\pi$ en $x_C = \frac{5}{2}\pi$. 1

$f\left(\frac{1}{2}\pi\right) = p$, $f\left(\frac{3}{2}\pi\right) = -p$ and $f\left(\frac{5}{2}\pi\right) = p$, so we have $A\left(\frac{1}{2}\pi, p\right)$, $B\left(\frac{3}{2}\pi, -p\right)$ and $C\left(\frac{5}{2}\pi, p\right)$. 1

$\overrightarrow{AB} = \begin{pmatrix} \pi \\ -2p \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} \pi \\ 2p \end{pmatrix}$. 1

These vectors have equal length for every value of p. 1

$\overrightarrow{AB} \cdot \overrightarrow{BC} = \pi^2 - 4p^2$

$$\begin{aligned} \overrightarrow{AB} \cdot \overrightarrow{BC} = 0 \implies & \pi^2 - 4p^2 = 0 \\ & 4p^2 = \pi^2 \\ & p^2 = \tfrac{\pi^2}{4} \\ & p = \tfrac{\pi}{2} \quad \text{(and } p = -\tfrac{\pi}{2} \text{ is not valid)} \end{aligned}$$ 1

Thus, there is one value of p for which triangle ABC is an isosceles right-angled triangle, namely $p = \frac{\pi}{2}$. 1