Practice exam mathematics A - solutions
[72 marks in total]

31 augustus 2021

**General:**

- sign error: −0.5 mark.

- The student did not write ′ = 0′ in an equation of the form \( ax^2 + bx + c = 0 \) before applying the quadratic formula: -0.5 marks for each time it occurs. (For example, a student has written \( 2x^2 + 3x - 4 \) without the ′ = 0′ behind it and instantly applies the quadratic formula).

- No worked-out solution = no marks
Exercise 1

a) [3 marks]

The given functions are \( f(x) = x^2 + 4x + 6 \) and \( g(x) = 1 - 2x \).

We have to solve \( f(x) = g(x) \) thus \( x^2 + 4x + 6 = 1 - 2x \) \( \Rightarrow \)

\[ x^2 + 6x + 5 = 0 \]

Quadratic formula:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
x = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 5}}{2}
\]

\[
x = \frac{-6 \pm \sqrt{36 - 20}}{2}
\]

\[
x = \frac{-6 \pm \sqrt{16}}{2}
\]

The intersection points of \( f \) and \( g \) are located at \( x = \frac{-6 + 4}{2} = -1 \) and \( x = \frac{-6 - 4}{2} = -5 \).

b) [2 marks]

\[
f(x) \cdot g(x) = (x^2 + 4x + 6) \cdot (1 - 2x)
\]

\[
= (x^2 + 4x + 6) + (-2x^3 - 8x^2 - 12x)
\]

\[
= x^2 + 4x + 6 - 2x^3 - 8x^2 - 12x
\]

\[
= -2x^3 - 7x^2 - 8x + 6
\]

c) [3 marks]

We have to determine the location of the extreme values of \( h(x) = f(x) \cdot g(x) \).

To this end we have to solve \( h'(x) = 0 \).

\( h'(x) = -6x^2 - 14x - 8 \).

Thus we solve \(-6x^2 - 14x - 8 = 0\)

Quadratic formula:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
x = \frac{14 \pm \sqrt{196 - 192}}{12}
\]

\[
x = \frac{14 \pm \sqrt{4}}{12}
\]

\[
x = \frac{14 \pm 2}{12}
\]

The \( x \) coordinates of the extreme values of \( h(x) \) are \( x = \frac{14 - 2}{12} = \frac{12}{12} = -1 \) and \( x = \frac{14 + 2}{12} = \frac{16}{12} = -\frac{4}{3} \).

d) [3 marks]

On the given interval \( g(x) > f(x) \). The distance function is therefore
\[ d(x) = g(x) - f(x) \]
\[ = (1 - 2x) - (x^2 + 4x + 6) \]
\[ = 1 - 2x - x^2 - 4x - 6 \]
\[ = -x^2 - 6x - 5 \]

We need to solve \( d'(x) = 0 \)
\[ d'(x) = -2x - 6 \]
So, we solve \(-2x - 6 = 0\).
Thus \(-2x = 6\)
\[ x = \frac{6}{-2} = -3 \]

Continue method 1 (one of the methods to continue has to be present in the written solution):
We see from the given graph that indeed there is a maximum in distance between the functions \( f \) and \( g \) at \( x = 3 \).

Continue method 2 (one of the methods to continue has to be present in the written solution):
\[ d''(x) = -2 \]
\[ d''(x) < 0 \] for all values \( x \), which means that there is a maximum in distance at \( x = 3 \).
Exercise 2

a) [1 mark]
The amount of duckweed doubles every three days. The growth factor per three days is therefore \( g_{3\text{ days}} = 2 \).

Now:
\[ g_{1\text{ day}} = g_{3\text{ days}} = 2^{\frac{1}{3}}. \]

b) [2 marks]
The general formula is:
\[ N(t) = N(0) \cdot g_{1\text{ day}}^t = 0.5 \cdot \left(2^{\frac{1}{3}}\right)^t = 0.5 \cdot 2^{\frac{1}{3}t}. \]

c) [3 marks]
In this case we have to solve:
\[ 13 = 0.5 \cdot 2^{\frac{1}{3}t} \text{ thus } \]
\[ \frac{13}{0.5} = 2^{\frac{1}{3}t} \]
\[ 26 = 2^{\frac{1}{3}t} \]
We apply the ground rule for exponents and logarithms: \( g^a = b \iff a = \log_g(b) \).
In this case \( g = 2, a = \frac{1}{3}t \) and \( b = 26 \), so
\[ \frac{1}{3}t = \log_2(26) \]
\[ t = 3 \cdot \log_2(26) \]
This cannot be simplified any further (it can be rewritten to \( 3 (\log_2(13) - \log_2(2)) = 3 (\log_2(13) - 1) = 3 \log_2(13) - 3 \) but this is not required).
After \( 3 \cdot \log_2(26) \) days an area of \( 13 \text{m}^2 \) of the pond is covered by duckweed.

d) [2 marks]
The decrease is 30\% a week. This means that 70\% of the former amount of duckweed is left after each week.
Thus \( g_{\text{week}} = 0.7 = \frac{7}{10} \).
The growth factor per two weeks is \( g_{\text{2 weeks}} = g_{\text{week}}^2 = \left(\frac{7}{10}\right)^2 = \frac{49}{100}. \)
Exercise 3

a) [6 marks]

The given function is \( f(x) = -2x^3 + x^2 + 4x \)

We have to determine the zeros, maxima, minima, domain, and range of this function.

For the zeros we solve \( f(x) = 0 \)

\[-2x^3 + x^2 + 4x = 0 \]
\[x(-2x^2 + x + 4) = 0 \]

Thus \( x = 0 \) or \(-2x^2 + x + 4 = 0\)

For the second equation we apply the quadratic formula:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
x = \frac{-1 \pm \sqrt{1^2 - 4(-2)(4)}}{2(-2)}
\]
\[
x = \frac{-1 \pm \sqrt{1 + 32}}{-4}
\]
\[
x = \frac{-1 \pm \sqrt{33}}{-4}
\]

The zeros of the function are therefore located at \( x = 0 \), \( x = \frac{-1 + \sqrt{33}}{4} \) and \( x = \frac{-1 - \sqrt{33}}{4} \).

For the extreme values we solve \( f'(x) = 0 \), after which we apply the second derivative criterion to determine the nature of the extreme value:

\[f'(x) = -6x^2 + 2x + 4 \]
\[f''(x) = -12x + 2 \]

Solving \( f'(x) = 0 \):

\[-6x^2 + 2x + 4 = 0 \]

Quadratic formula:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
x = \frac{-2 \pm \sqrt{2^2 - 6(4)}}{-12}
\]
\[
x = \frac{-2 \pm \sqrt{4 + 24}}{-12}
\]
\[
x = \frac{-2 \pm \sqrt{28}}{-12}
\]
\[
x = \frac{-2 \pm 2\sqrt{7}}{-12}
\]

Thus \( x = \frac{1}{6} + \frac{\sqrt{7}}{6} = \frac{5}{6} = 1 \) or \( x = \frac{1}{6} - \frac{\sqrt{7}}{6} = -\frac{4}{3} = -\frac{2}{3} \).

Now we determine which of these is a maximum and which is a minimum:

\[f''(1) = -12 \cdot 1 + 2 = -10 < 0 \text{ which means this is a maximum} \]
\[f''\left(-\frac{2}{3}\right) = -12 \cdot -\frac{2}{3} + 2 = 24 \cdot \frac{3}{3} + 2 = 8 + 2 = 10 > 0 \text{ which means this is a minimum} \]

Now we determine the domain and range:

There are no restrictions for substituting values for \( x \) in this function. The domain is \( \mathbb{R} \). (can also be written as: \( (-\infty, \infty) \))

This function does not have asymptotes or global minima/maxima. All values of \( y \) are reached. The range is the full \( \mathbb{R} \). (can also be written as: \( (-\infty, \infty) \))
b) [5 marks]
The given function is \( g(x) = \frac{4x-5}{10-3x} \).
We have to determine the zeros, maxima, minima, domain, and range of this function.

For the zeros, we solve \( g(x) = 0 \):

\[
\frac{4x-5}{10-3x} = 0
\]

\[
10 - 3x = 0
\]

\[
x = \frac{5}{3}
\]

This function has a zero at \( x = \frac{5}{3} \).

For the extreme values, we solve \( g'(x) = 0 \), after which we apply the second derivative criterion to determine the nature of the extreme value:

\[
g'(x) = \frac{d}{dx} \left( \frac{4x-5}{10-3x} \right)
\]

\[
= \frac{(10-3x)(4) - (4x-5)(-3)}{(10-3x)^2}
\]

\[
= \frac{40 - 12x + 12x - 15}{(10-3x)^2}
\]

\[
= \frac{25}{(10-3x)^2}
\]

(the second derivative does not have to be determined in this case)

We solve \( g'(x) = 0 \) thus \( \frac{25}{(10-3x)^2} = 0 \).

This would imply that \( 25 = 0 \), which is not possible. There are no solutions.

We conclude that this function does not have extreme values.

(You can also conclude that a linear fractional function does not have extreme values and that this implies that there are no solutions to \( g'(x) = 0 \), but this must be written in a clear and neat way).

We determine the domain and range:

For the domain, we determine the vertical asymptote of \( g(x) \):
This is the value of \( x \) which cannot be substituted into the function.
The denominator cannot be zero, therefore we solve:

\[
10 - 3x = 0
\]

\[
-3x = -10
\]

\[
x = \frac{10}{3}
\]

This means that the domain of the function is \( \mathbb{R} \setminus \left\{ \frac{10}{3} \right\} \).

- This can also be written as: \((-\infty, \frac{10}{3})\) and \((\frac{10}{3}, \infty)\).

For the range, we determine the horizontal asymptote.
We need to divide the coefficient of \( x \) in the numerator by the coefficient of \( x \) in the denominator. This gives:

\[
y_{asym} = \frac{4}{-3} = -\frac{4}{3}.
\]

Therefore, the range is \( \mathbb{R} \setminus \left\{ -\frac{4}{3} \right\} \).
• This can also be written as: \((-\infty, -\frac{4}{3})\) and \(\left(\frac{4}{3}, \infty\right)\).
Exercise 4

a) [4 marks]

\[
\begin{align*}
\sqrt{6x - 2} &= 2x \\
6x - 2 &= (2x)^2 \\
6x - 2 &= 4x^2 \\
4x^2 - 6x + 2 &= 0
\end{align*}
\]

We apply the quadratic formula:
(Determining the determinant firstly is correct as well)

\[
x = \frac{6 \pm \sqrt{(-6)^2 - 4 \cdot 4 \cdot 2}}{8} = \frac{6 \pm \sqrt{36 - 32}}{8} = \frac{6 \pm \sqrt{4}}{8}
\]

Thus \( x = \frac{6 + 2}{8} = 1 \) and \( x = \frac{6 - 2}{8} = \frac{1}{2} \).

Now we have to check these solutions by substituting them into the left-hand side and right-hand side of the original equation:

The solution \( x = 1 \):

- Left: \( \sqrt{6 \cdot 1 - 2} = \sqrt{4} = 2 \).
- Right: \( 2 \cdot 1 = 2 \).
- Left = Right, so this is a solution to the original equation.

The solution \( x = \frac{1}{2} \):

- Left: \( \sqrt{6 \cdot \frac{1}{2} - 2} = \sqrt{3 - 2} = \sqrt{1} = 1 \).
- Right: \( 2 \cdot \frac{1}{2} = 1 \).
- Left-hand = Right-hand, so this is a solution to the original equation.

The solutions of the equation are therefore \( x = 1 \) and \( x = \frac{1}{2} \).

b) [4 marks]

Method 1:

We apply the ground rule for exponents and logarithms.

When \( g^a = b \iff a = \log_g(b) \)
\[ 7^{1-x} = \sqrt{7} \]
\[ 1 - x = \log_7 \left(7^{\frac{1}{2}}\right) \]
\[ = \frac{1}{2} \log_7 (7) \]
\[ = \frac{1}{2} \]

\[-x = -\frac{1}{2} \]
\[ x = \frac{1}{2} \]

**Method 2:**

\[ 7^{1-x} = \sqrt{7} \]
\[ 7^{1-x} = 7^{\frac{1}{2}} \]

\[ 1 - x = \frac{1}{2} \]
\[ -x = -\frac{1}{2} \]
\[ x = \frac{1}{2} \]
Exercise 5 [5 marks]

A sine wave with amplitude $A$, equilibrium $B$, frequency $f$ and phase $\phi$ is written as:

$$s(t) = B + A \cdot \sin(2\pi \cdot f \cdot t + \phi)$$

The period $T$ is given by $T = \frac{1}{f}$ and thus the frequency is $f = \frac{1}{T}$. In addition, the coordinates of a starting point are given by

$$(t, s(t)) = \left( -\frac{\phi}{2\pi f}, B \right)$$

We start by determining the amplitude. The amplitude is equal to half of the distance, in the $y$ direction, between two extreme values. In this case we see from the graph that the function values range from $-6$ up until $2$. The distance in $y$ direction between the extreme values is therefore $8$. Therefore, the amplitude equals $A = \frac{8}{2} = 4$.

The equilibrium $B$ lies on the mean value between $-6$ end $2$, and therefore equals $B = \frac{-6+2}{2} = -2$.

To determine the frequency we first read the period, after which we use $f = \frac{1}{T}$. We see that the function has a minimum at $t = \frac{1}{3}$ and at $t = \frac{13}{3}$. Thus the period is $T = \frac{13}{3} - \frac{1}{3} = \frac{12}{3} = 4$. Which means that the frequency is $f = \frac{1}{4}$.

The factor $2\pi f = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$.

To determine the phase $\phi$ we use a starting point. In this case you can choose the starting point located at $t = \frac{4}{3}$ or the one located at $t = \frac{16}{3}$. Both are correct (you just get another value for $\phi$). In this solution we will work with the one located at $t = \frac{4}{3}$. This implies that:

$$\frac{-\phi}{2\pi f} = \frac{4}{3}$$

$$\frac{-\phi}{\left(\frac{\pi}{2}\right)} = \frac{4}{3}$$

$$-\phi = \frac{4 \cdot \pi}{3} \cdot \frac{2}{2}$$

$$-\phi = \frac{2\pi}{3}$$

$$\phi = -\frac{2\pi}{3}$$

We substitute the values of the variables into the general formula for a sine wave (and simplify the factor $2\pi f$):

$$s(t) = -2 + 4 \cdot \sin \left( \frac{1}{2} \pi t - \frac{2\pi}{3} \right).$$
Exercise 6

a) [4 marks]

Method 1:

\[ f(x) = \log_3(2x) - 4 \]

\[
\begin{align*}
  f'(x) &= (\log_3(2x))' - (4)' \\
        &= \left(\frac{\ln(2x)}{\ln(3)}\right)' - 0 \\
        &= \frac{1}{2x \cdot \ln(3)} \cdot 2x \\
        &= \frac{2 \ln(3)}{x} \\
        &= \frac{2x \ln(3)}{x \ln(3)}
\end{align*}
\]

The last step in gray is not required.

Method 2:

Substitute \( u = 2x \)

\[
\begin{align*}
  f'(x) &= (\log_3(2x))' - (4)' \\
        &= \left(\frac{\ln(u)}{\ln(3)}\right)' \cdot u' \\
        &= \frac{1}{u \ln(3)} \cdot 2 \\
        &= \frac{2 \ln(3)}{x} \\
        &= \frac{2x \ln(3)}{x \ln(3)}
\end{align*}
\]

The last step in gray is not required.

b) [2 marks]

\( g(x) = x\sqrt{x} \)

Method 1:

\[ g(x) = x\sqrt{x} = x \cdot x^{\frac{1}{2}} = x^{\frac{3}{2}} \]

Thus:

\[
\begin{align*}
  g'(x) &= \frac{3}{2} x^{\frac{3}{2} - 1} \\
        &= \frac{3}{2} x^{\frac{1}{2}}
\end{align*}
\]

Method 2: (using the product rule):

\[
\begin{align*}
  g'(x) &= (x)' \cdot \sqrt{x} + x \cdot (\sqrt{x})' \\
        &= \frac{1}{2} x^{-\frac{1}{2}} \sqrt{x} + x \cdot \left(\frac{1}{2} x^{-\frac{1}{2}}\right)' \\
        &= \sqrt{x} + \frac{1}{2} x^{-\frac{1}{2}} \\
        &= \frac{3}{2} x^{\frac{1}{2}}
\end{align*}
\]
Exercise 7 [4 marks]

This is a geometric sequence.
In this case, \( k \) does not start at 0, but at 3.
Let \( b = k - 3 \), and thus \( k = b + 3 \). Then:

\[
\sum_{k=3}^{8} 3 \cdot 2^k = \sum_{b=0}^{5} 3 \cdot 2^{b+3}
\]
\[
= \sum_{b=0}^{5} 3 \cdot 2^3 \cdot 2^b
\]
\[
= \sum_{b=0}^{5} 3 \cdot 8 \cdot 2^b
\]
\[
= \sum_{b=0}^{5} 24 \cdot 2^b
\]

Now we have a geometric sequence starting at \( b = 0 \) with \( a_0 = 24 \), \( r = 2 \) and \( n = 5 \) thus \( n + 1 = 6 \). Applying the sum formula:

\[
\sum_{b=0}^{5} 24 \cdot 2^b = \frac{24 (1 - 2^6)}{1 - 2}
\]
\[
= \frac{24 (1 - 64)}{1 - 2}
\]
\[
= \frac{-24 \cdot -63}{-1}
\]
\[
= 24 \cdot 63
\]

= After this step using a calculator is allowed, or (gray steps):
= \( 10 \cdot 63 + 10 \cdot 63 + 4 \cdot 63 \)
= \( 630 + 630 + 252 \)
= \( 1260 + 252 \)
= \( 1512 \)

The sum is therefore 1512.
Exercise 8

a) [3 marks]
We have 25 lemons and 5 limes. The total number of fruits is therefore 30. We grab five times. Requested is the probability that we grab 5 limes.

\[ P(5 \text{ limes}) = \frac{\text{Number of possible sequences}}{\text{Number of sequences}} \times P(\text{one sequence}) \]

The number of possible sequences for 5 limes is: 1.
The probability for this one sequence is:

\[ P(LLLLL) = \frac{5}{30} \cdot \frac{4}{29} \cdot \frac{3}{28} \cdot \frac{2}{27} \cdot \frac{1}{26} \]

Thus:

\[ P(5 \text{ limes}) = 1 \cdot \frac{5}{30} \cdot \frac{4}{29} \cdot \frac{3}{28} \cdot \frac{2}{27} \cdot \frac{1}{26} \]

You can leave this as it is.

b) [3 marks]
Now we need to calculate the probability we grab more than 3 limes.
This means the probability \( P(4 \text{ limes OR 5 limes}) \)

\[ P(4 \text{ limes OR 5 limes}) = P(4 \text{ limes}) + P(5 \text{ limes}) \]

In item (a) we already determined the probability to grab 5 limes. So we only have to determine the probability to grab 4 limes.

\[ P(4 \text{ limes}) = \frac{\text{Number of possible sequences}}{\text{Number of sequences}} \times P(\text{one sequence}) \]

The number of possible sequences for 4 limes and 1 lemon is:

\[ \frac{5!}{4!1!} \left( \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 1} = 5 \right) \]

The probability of one of these sequences is:

\[ P(LLLLC) = \frac{5}{30} \cdot \frac{4}{29} \cdot \frac{3}{28} \cdot \frac{2}{27} \cdot \frac{25}{26} \]

Thus:

\[ P(5 \text{ limes}) = \frac{5!}{4!1!} \cdot \frac{5}{30} \cdot \frac{4}{29} \cdot \frac{3}{28} \cdot \frac{2}{27} \cdot \frac{25}{26} \]

Therefore

\[ P(4 \text{ limes OR 5 limes}) = \frac{5!}{4!1!} \cdot \frac{5}{30} \cdot \frac{4}{29} \cdot \frac{3}{28} \cdot \frac{2}{27} \cdot \frac{25}{26} + \frac{5!}{4!1!} \cdot \frac{5}{30} \cdot \frac{4}{29} \cdot \frac{3}{28} \cdot \frac{2}{27} \cdot \frac{1}{26} \]
Exercise 9 [2 marks]

We have 20 tiles in five different colours (4 tiles of each color). The edge of tile on a wall also has a length of 20 tiles, which means we will use all the tiles. Furthermore, tiles of the same colour may sit next to each other. This implies that we have to calculate the number of ways in which we can order 20 tiles. There are five groups of 4 tiles each which cannot be distinguished from each other.

The number of sequences is therefore:

\[
\frac{20!}{4!4!4!4!4!}
\]
Exercise 10 [2 marks]

The test consists of 20 questions with 4 choices each. You guess all twenty questions. We have to calculate the probability that you answered half of the questions correctly, thus $P(10 \text{C})$.

This problem corresponds to a binomial distribution (using an urn model is possible as well but will be more complicated).

For each question there is a $\frac{1}{4}$ chance to guess it correctly.

Now, call success = guessed correctly, then:

- $n = 20$
- $p = \frac{1}{4}$
- $k = 10$

Thus

$$P(10 \text{C}) = \binom{20}{10} \left(\frac{1}{4}\right)^{10} \left(\frac{3}{4}\right)^{10}$$

It is also correct to write this as:

$$P(10 \text{C}) = \binom{20}{10} 0.25^{10} 0.75^{10}$$
Exercise 11

a) [4 marks]

We have to calculate \( P(X > 335) \).

\[
P(X > 335) = 1 - P(X < 335)
\]

The value of 335 is located above the mean value, therefore we can apply the transformation rule for a normal distribution:

\[
P(X > 335) = 1 - P(Z < \frac{335 - 330}{10})
\]

\[
= 1 - P(Z < 0.5)
\]

\[
= 1 - 0.6915
\]

\[
= 0.3085
\]

\[
= 0.309
\]

b) [4 marks]

Now we need to determine the following probability:

\[
P(X < 315 \text{ or } X > 345) = P(X < 315) + P(X > 345)
\]

\[
= P(X > 345) + P(X > 345)
\]

\[
= 2 \cdot P(X > 345)
\]

\[
= 2 \cdot (1 - P(X < 345))
\]

\[
= 2 \cdot (1 - P(Z < \frac{345 - 330}{10}))
\]

\[
= 2 \cdot (1 - P(Z < 1.5))
\]

\[
= 2 \cdot (1 - 0.9332)
\]

\[
= 2 \cdot 0.0668
\]

\[
= 0.1336
\]

\[
= 0.134
\]