Michel Mandjes & Onno Boxma

The Cramér-Lundberg model and its variants

A queueing perspective

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Preface

This book is about the most intensively studied model in ruin theory, usually referred to as the *Cramér-Lundberg* model, and several of its variants. In its most basic form, the Cramér-Lundberg model was introduced in the early 1900s, in order to assess an insurance company's vulnerability to ruin. While the model should be considered as a stylized description of reality, it nicely captures the essentials of the dynamics of the insurance firm's surplus level, which explains why it has become the undisputed benchmark model in ruin theory.

Over the past century many techniques have been developed to analyze the Cramér-Lundberg model, predominantly focusing on quantifying the probability of ruin. The objective of this book is to provide an account of the main results and underlying probabilistic techniques. Wherever this is useful, we exploit connections with associated models from queueing theory, a related discipline within the area of applied probability. The ambition has been to present the material in a maximally compact, systematic, transparent and consistent manner.

The target audience of the book are master students, and (starting) PhD students, in (applied) mathematics, operations research and actuarial science. Regarding prior knowledge, we have tried to make the material as accessible and self-contained as possible. This concretely means that in principle only a relatively basic background in probability theory is needed — an introductory probability course and a Markov chains course should suffice. This being said, some familiarity with working with transforms is of great help. In addition, in a few places more sophisticated probabilistic concepts play a role, in particular martingales, but only basic knowledge of those notions is required. We finally mention that there are a few proofs in which results from complex analysis play a role.

The book is particularly suited for a semester-long course. A collection of slides can be found on Michel Mandjes' personal website, where in addition a list of errors and typos will be kept updated. The material is accompanied by a carefully selected set of exercises. These are intended to deepen the students' knowledge and understanding of the course material. There is some diversity in terms of the level of difficulty — the exercises marked with the symbol ^{1}¢^{o} can be considered as rather challenging. As the first chapter is of a more methodological nature, laying the foundations for many of the derivations later in the book, it is advised to allocate considerably more time to it than to the other chapters. Also, some specific sections are relatively advanced: this in particular applies to Sections 7.5, 8.4, 10.3, and 10.4, which may be left out when teaching from the book. We included two appendices to provide the reader with some additional background on concepts that play a role throughout the book: the first appendix concerns Laplace transforms and Laplace-Stieltjes transforms, while the second appendix covers a collection of basic results from queueing theory.

While we have tried to be as consistent as possible in terms of notation, in a textbook of this size it is inevitable that sometimes the same symbol is used for different quantities. Within chapters such conflicts of notation have largely been avoided, but between the chapters there are a few such inconsistencies. We do believe, though, that the local meaning of all symbols is clear from the context.

While writing the book we reviewed a substantial body of existing work on Cramér-Lundberg-type models. A happy consequence was that along the way we managed to derive various new results. Most notably, novel results contained in the book are: (1) the full analysis of the two-level Cramér-Lundberg model of Chapter 5, (2) the analysis of the transient behavior in the finite-workload M/G/1 queue, as covered by Exercise 5.5, (3) the results on a specific level-dependent model of Section 6.4, (4) the derivation of the Gerber-Shiu metrics for the multivariate ruin problem, presented in Section 7.5, (5) the results on Poisson inspection schemes of Section 10.2, in particular the decomposition of Theorem 10.2, and (6) the analysis of various more involved bankruptcy concepts, as addressed in Sections 10.3–10.4.

Compared to the related textbook [1] our monograph has a somewhat stronger focus on a broad range of relevant non-standard variants of the Cramér-Lundberg model, explicitly exploiting the relation with queueing theory; compared to the comprehensive textbooks [2, 3, 5, 6], our monograph provides a very detailed account of ruin, and our analysis is often using elements from queueing theory. The book [7] considers the setting of claims arriving according to a renewal process, while in our book we predominantly focus on the case of Poisson claim arrivals (as is the case in the Cramér-Lundberg model). In [4] a set of specific topics on advanced actuarial models is discussed. Another distinguishing element is that throughout our book there is a strong focus on ruin before an exponentially distributed time, uniquely characterizing the time-dependent ruin probability. In general, also with our target audience in mind, our focus has been on explaining the main ideas behind model variants and techniques, rather than on providing a complete account of the existing literature. Each chapter is concluded with a discussion and bibliographical notes, putting the results of the chapter into perspective and providing a (non-exhaustive) set of relevant references. Our book also contains a number of short biographies of influential probabilists and actuarial scientists, at the end of Chapter 1.

The book consists of ten chapters, each of them focusing on a specific aspect or model variant. Chapter 1 provides a detailed introduction of the conventional

Cramér-Lundberg model, including its relation to the M/G/1 queue. Four techniques are discussed that provide us with an explicit expression for the transform of the ruin probability over an exponentially distributed time horizon. In Chapter 2 again the Cramér-Lundberg model is considered in its most basic form, but now with a focus on the asymptotics of the ruin probability in the regime that the insurance firm's initial reserve level grows large.

The other eight chapters focus on the analysis of the ruin probability for a set of natural variants of the classical Cramér-Lundberg model. In Chapter 3 an analysis is provided for the Cramér-Lundberg model under regime switching, entailing that the specifics of the reserve-level process are affected by an autonomously evolving Markovian background process. Chapter 4 considers a model that also includes the interest received by the insurance firm and in addition allows positive jumps in the reserve-level process. Chapter 5 considers a setting where the reserve-level process behaves differently above resp. below a certain threshold. This idea is then further developed in Chapter 6, where the model parameters depend on the reserve level in a continuous manner. Chapter 7 considers ruin in a multivariate context, under a specific ordering condition that is imposed on the marginal reserve-level processes. While in the rest of the monograph claim arrival processes are assumed to be of the Poisson-type, Chapter 8 considers the effect of various types of clustering in the claim arrival process on the ruin probability. Then, in Chapter 9, we analyze several models in which there is dependence between claim sizes and claim interarrival times. Finally, Chapter 10 provides an analysis of various sophisticated bankruptcy concepts.

We recommend to start by reading Chapter 1, as it describes various objects and concepts that are prerequisites for the analysis presented in the other chapters. Chapters 2–10 can in principle be read independently of each other and essentially in any order. A few exceptions are: (i) in order to understand the change-of-measure based analysis in Section 5.2 and Chapter 8, it helps to first read Section 2.2, (ii) it is helpful to read Chapter 6 with the results of Chapter 3 in mind.

We wish to thank everybody who has helped improving the quality of the book. In this respect it is important to mention that the book has been used in a 'test run' in the master program *Stochastics & Financial Mathematics* at the University of Amsterdam in the spring semesters of 2022 and 2023. The presentation of the book has benefited substantially from the students' feedback. We would like to thank (then) master students Riccardo Alberti, Saaly Alsaadi, Wouter Andringa, Rens Blaauwendraad, Konstantinos Chatziandreou, Nefeli Groenheijde-Rousta, Teyssir Hilmi, Petar Jerončić, Florine Kuipers, Sara Morcy, Tygo Nijsten, Pieter Out, Livitsanos Vasillios, and Chris Vermeulen, but in particular Fabian Hinze, Cecille Hossainkhan, and Valerii Zoller, for their careful reading of the manuscript, their willingness to 'test' exercises, and their helpful comments in general.

In addition, the book has been read by our colleagues Esther Frostig, Zbigniew Palmowski, and Jacques Resing, whose valuable feedback is greatly appreciated. Our interest in this class of models was triggered through inspiring joint research projects with various colleagues — in this context we would like to explicitly mention

Hansjörg Albrecher, Krzysztof Dębicki, Peter Glynn, Offer Kella, and Josh Reed. Hansjörg Albrecher in addition kindly gave us valuable feedback on the manuscript, while also providing useful advice regarding publishing this book in the Springer Actuarial Series.

Amsterdam, Eindhoven, *Michel Mandjes*
June 13, 2023 *Onno Boxma*

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Contents

Chapter 1
Cramér-Lundberg model

Abstract — In this chapter we discuss the conventional Cramér-Lundberg ruin model. The focus lies on evaluating transforms related to the all-time and time-dependent ruin probabilities (where the latter, in our setting, concerns ruin before an exponentially distributed amount of time). An important role is played by a duality with the M/G/1 queueing model. We present four independent analysis techniques; they differ in the sense whether the ruin model or the corresponding queueing model (or a mixture of both) has been used.

1.1 Introduction

Arguably the most frequently used model in ruin theory is the *Cramér-Lundberg model*. In this model independent and identically distributed (i.i.d.) claims are generated according to a Poisson process, whereas the insurance firm receives premiums at a constant rate. The key object of study is the *ruin probability*, i.e., the probability that for a given initial reserve, the reserve level drops below zero. Ruin probabilities in this book come in two flavors: the all-time and time-dependent ruin probability. The former concerns ruin over an infinite time horizon, and the latter aims at characterizing the probability of ruin before a given time.

In the analysis one often works with the *net cumulative claim process*. This process records the cumulative amount of claimed money, decreased by the premiums earned. The insurance firm is ruined when this net cumulative claim process exceeds the initial reserve. A consequence of this observation is that the insurance firm's ruin can be written in terms of the running maximum process (corresponding to the net cumulative claim process, that is) exceeding a given threshold (the initial reserve, that is).

In this monograph, we intensively use a duality relation between the event of ruin in the Cramér-Lundberg model, and the event of a workload threshold being exceeded in a related queueing model. This related queueing model, which is called the M/G/1 model in the queueing literature, has been widely studied, sometimes using methods which are less common in insurance mathematics, thus opening up the possibility of using methods and results for the M/G/1 queue when analyzing ruin probabilities. We refer to Appendix B for a brief account of the M/G/1 queue.

In Section 1.2 we start by formally defining the Cramér-Lundberg model and establishing the duality relation with the M/G/1 queue. Then, in Sections 1.3–1.6, we detail four methods by which we can analyze the transform of the ruin probability. These methods differ in the extent to which in the argumentation the ruin model and the queueing model have been used. In more detail, the following approaches will be presented:

In the first approach we exclusively use the ruin model. We translate the quantities of our interest in terms of the Laplace-Stieltjes transform of the running maximum of the net cumulative claim process. Explicit expressions are found by conditioning on the first event, which corresponds to either a claim arrival or to having reached the time horizon.

The second approach uses both the ruin model and the queueing model. The main underlying idea is to write the running maximum as the sum of a geometric number of i.i.d. random quantities, which are interpreted as so-called ladder heights. In the derivation of the distribution of these ladder heights (in terms of their Laplace transform), queueing-theoretic results are heavily relied upon.

A third approach works with the queueing representation, and relies on the so-called Kella-Whitt martingale. By applying the optional sampling machinery, the transform of the quantity of interest allows direct evaluation.

In the fourth approach, we again use the queueing representation. By carefully evaluating the probabilistic dynamics of the workload, we set up a system of differential equations for the transform under study, which can be evaluated explicitly. In the queue's stationary regime an appealing rate-conservation interpretation is given.

At the end of this chapter, in Section 1.8, we have included biographical sketches of six researchers who had a crucial impact on the material covered in this book.

1.2 Ruin model, and dual queueing model

In the conventional Cramér-Lundberg framework, claims arrive according to a Poisson process with rate $\lambda > 0$. The claims form a sequence of i.i.d. random variables $B_1, B_2, \ldots$, distributed as a generic non-negative random variable B whose Laplace-Stieltjes transform is, for $\alpha > 0$, given by

$$b(\alpha) := \mathbb{E}\, e^{-\alpha B} = \int_{[0,\infty)} e^{-\alpha x}\, \mathbb{P}(B \in \mathrm{d}x).$$

We refer to Appendix A for an extensive exposition of transforms, with a specific focus on the Laplace-Stieltjes transform (LST, an abbreviation that is consistently used throughout the monograph). The clients pay premiums, i.e., income for the insurance company, at a constant rate $r > 0$. The initial surplus level is $u > 0$.

Let $N(t)$ denote the number of claims that have arrived in $[0, t]$. Due to the exponentially distributed inter-claim times, $N(t)$ has a Poisson distribution with mean λt. Until ruin, the surplus level at time t is given by

$$X_u(t) := u + rt - \sum_{i=1}^{N(t)} B_i,$$

following the convention that an empty sum is defined as 0.

∘ *Ruin probabilities.* The first objective is to characterize the (infinite-time) ruin probability, given that the initial reserve level is u, i.e., the probability of $X_u(t)$ ever dropping below 0. It is formally defined by

$$p(u) := \mathbb{P}(\exists s > 0 : X_u(s) < 0).$$

In specific cases, it is convenient to work with the complement of the ruin probability; this *survival probability* is defined as $\bar{p}(u) := 1 - p(u)$.

In the second place, we analyze its time-dependent counterpart, i.e.,

$$p(u, t) := \mathbb{P}(\exists s \in [0, t] : X_u(s) < 0).$$

Remark 1.1 Evidently, the probability $p(u)$ is trivially 1 if the so-called *net-profit condition* $\lambda\,\mathbb{E}B < r$ is violated; here $\lambda\,\mathbb{E}B$ can be interpreted as the expected claimed amount per time unit, while r is the insurer's income per time unit. The probability $p(u, t)$ is worth studying regardless of whether or not the condition $\lambda\,\mathbb{E}B < r$ is in place. ◇

Only in exceptional cases, the ruin probability $p(u, t)$ allows a closed-form expression. This explains why one often works with transforms. It concretely means that we settle for developing techniques to compute the Laplace transform (LT, an abbreviation used throughout the monograph; again, see Appendix A) of the infinite-time ruin probability $p(\cdot)$, i.e.,

$$\pi(\alpha) := \int_0^\infty e^{-\alpha u}\, p(u)\, \mathrm{d}u,$$

and its time-dependent counterpart. In the latter case, for reasons of tractability we predominantly focus on an exponentially distributed time horizon. This concretely means that, with T_β an exponentially distributed time with mean β^{-1} (independent of the process $X_u(t)$), we consider the LT of the ruin probability $p(u, T_\beta)$, corresponding to ruin before T_β. In other words, we focus on evaluating the double transform

$$\pi(\alpha, \beta) := \int_0^\infty \int_0^\infty \beta\, e^{-\alpha u - \beta t}\, p(u, t)\, \mathrm{d}u\, \mathrm{d}t$$

for $\alpha > 0$ and $\beta > 0$. Relying on a standard Abelian theorem, we have that $\varphi(\alpha) = \lim_{\beta\downarrow 0}\varphi(\alpha,\beta)$, so that it suffices to focus on evaluating $\varphi(\alpha,\beta)$ only. In the sequel the exponentially distributed 'clock' T_β is often referred to as the *killing epoch*.

B *Transform of running maximum.* We define the 'net cumulative claim process' $Y(t) := u - X_u(t)$ and the corresponding running maximum process $\bar{Y}(t)$:

$$Y(t) := \sum_{i=1}^{N(t)} B_i - rt, \qquad \bar{Y}(t) := \sup_{s\in[0,t]} Y(s).$$

In the literature, the process $Y(t)$ is often referred to as a *compound Poisson process with drift*. Notice that, clearly,

$$p(u) = \mathbb{P}(\exists s > 0 : X_u(s) < 0) = \mathbb{P}(\bar{Y}(\infty) > u), \qquad p(u,t) = \mathbb{P}(\bar{Y}(t) > u).$$

This shows that the probabilities $p(u)$ and $p(u,t)$ can (as functions of $u > 0$, that is) be interpreted as the complementary cumulative distribution functions of the random variables $\bar{Y}(\infty)$ and $\bar{Y}(t)$, respectively. See Figure 1.1 for a realization of the net cumulative claim process and the corresponding reserve-level process.

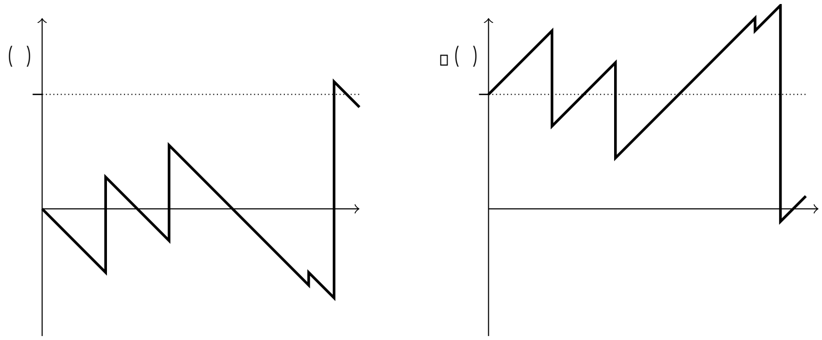

Fig. 1.1 Net cumulative claim process $Y(t)$ (left panel) and the corresponding reserve-level process $X_u(t)$ (right panel).

Remark 1.2 Consider the evaluation of the object

$$r(\alpha,\beta) := \mathbb{E}\, e^{-\alpha \bar{Y}(T_\beta)} = \int_0^\infty e^{-\alpha u}\, \mathbb{P}(\bar{Y}(T_\beta) \in \mathrm{d}u).$$

By applying integration by parts, $r(\alpha,\beta)$ can be expressed in terms of $\varphi(\alpha,\beta)$, as follows:

$$r(\alpha,\beta) = \int_0^\infty e^{-\alpha u}\, \mathrm{d}\mathbb{P}(\bar{Y}(T_\beta) > u)$$

$$= -\left[e^{-\alpha u}\,\mathbb{P}(\bar{X}(T_\beta) > u)\right]_{u=0}^{\infty} - \alpha\int_0^\infty e^{-\alpha u}\,\mathbb{P}(\bar{X}(T_\beta) > u)\,\mathrm{d}u$$

$$= 1 - \alpha\int_0^\infty e^{-\alpha u}\,p(u, T_\beta)\,\mathrm{d}u = 1 - \alpha\,\pi(\alpha, \beta), \tag{1.1}$$

cf. Appendix A.2. From the above we conclude that, when aiming at computing $\pi(\alpha, \beta)$, we can equivalently compute $r(\alpha, \beta)$, in the sense that these two double transforms uniquely define one another. ◇

Remark 1.3 The probability $p(u, T_\beta)$ has the following alternative interpretation. Define by $\tau(u)$ the first time that the process $X(t)$ enters the set $[u, \infty)$, i.e., $\tau(u)$ denotes the *ruin time*:

$$\tau(u) := \inf\{t \geq 0 : Y_u(t) \leq 0\} = \inf\{t \geq 0 : X(t) \geq u\}.$$

Then it is easily verified (e.g. by conditioning on the value of $\tau(u)$) that

$$p(u, T_\beta) = \mathbb{P}(\tau(u) < T_\beta) = \mathbb{E}\, e^{-\beta\tau(u)}1\{\tau(u) \leq \infty\}, \tag{1.2}$$

where $1\{E\}$ is the indicator function of the event E. We thus conclude that the evaluation of the ruin probability over an exponentially distributed horizon is equivalent to the evaluation of the Laplace transform of the ruin time $\tau(u)$ (on the event of this ruin time being finite). For more background on this interpretation of the LST as a probability, see Appendix A.2. Noting that

$$p(u) = \mathbb{P}(\tau(u) \leq \infty) = \lim_{\beta\downarrow 0}\mathbb{P}(\tau(u) < T_\beta) = \lim_{\beta\downarrow 0}\mathbb{E}\, e^{-\beta\tau(u)}1\{\tau(u) \leq \infty\},$$

the all-time ruin probability can be evaluated as well. ◇

∘ *Duality with M/G/1 queue.* An M/G/1 queue models a reservoir at which i.i.d. jobs (distributed as a generic non-negative random variable B) arrive according to a Poisson process with rate $\lambda > 0$, while the system is drained at a deterministic rate $r > 0$. The workload in this system, denoted by $Q(t)$ at time $t \geqslant 0$, can be seen as the net input process $X(t)$ truncated at zero (thus preventing the storage level from becoming negative).

We now establish an important duality relation between the workload $Q(t)$ in the M/G/1 model on one hand, and the running maximum $\bar{X}(t)$ of our net cumulative claim process $X(t)$ on the other hand, where we throughout assume that $Q(0) = 0$. To this end, define the running minimum process by

$$\underline{X}(t) := \inf_{s\in[0,t]} X(s).$$

Considering Figure 1.2, upon inspecting both panels, we conclude the validity of the key identity

$$Q(t) = X(t) - \underline{X}(t).$$

Here the increasing process $-\underline{X}(t)/r$ can be interpreted as the cumulative time that the workload process $Q(t)$ spends at zero, i.e., the queue's cumulative idle time. In addition the following distributional equality applies: relying on a time-reversibility argument,

$$X(t) - \underline{X}(t) = X(t) - \inf_{s\in[0,t]} X(s) = \sup_{s\in[0,t]} (X(t) - X(s))$$
$$\stackrel{\rm d}{=} \sup_{s\in[0,t]} X(s) = \bar{X}(t),$$

with '$\stackrel{\rm d}{=}$' denoting equality in distribution. We thus conclude that $\bar{X}(t)$ has the same distribution as $Q(t)$. We will extensively use this duality property throughout this monograph.

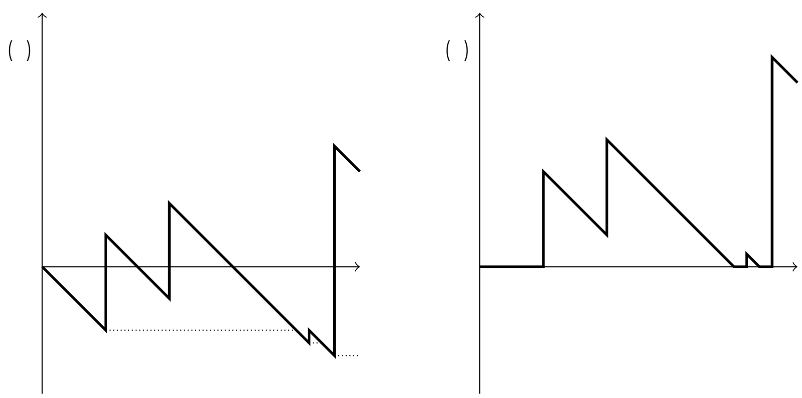

Fig. 1.2 Net cumulative claim process $X(t)$ (left panel) and workload process $Q(t)$ (right panel) for a compound Poisson process. In the left panel, the corresponding running minimum process $\underline{X}(t)$ is depicted by the dotted lines.

1.3 Method 1: conditioning on the first event

This section contains a first way to compute our target quantity $\kappa(\alpha,\beta)$. The main underlying idea is that we evaluate $\kappa(\alpha,\beta)$ by conditioning on the first event, which is either a claim arrival or killing. We thus obtain an expression in terms of the transform of interest $\kappa(\alpha,\beta)$, which we can then solve from the resulting equation (also requiring the identification of an unknown constant).

As a first step, recalling that T_β is exponentially distributed with mean β^{-1}, and that hence with probability $\lambda/(\lambda+\beta)$ the first claim arrival occurs before killing, we write

$$p(u,T_\beta) = \frac{\lambda}{\lambda+\beta}\big(p_1(u,T_\beta) + p_2(u,T_\beta)\big),$$

where, distinguishing between the scenario that there is ruin due to the first claim and the scenario that multiple claims are needed,

$$p_1(u,T_\beta) := \int_0^\infty (\lambda+\beta)\,e^{-(\lambda+\beta)s}\int_{u+rs}^\infty \mathbb{P}(B\in \mathrm{d}v)\,\mathrm{d}s,$$
$$p_2(u,T_\beta) := \int_0^\infty (\lambda+\beta)\,e^{-(\lambda+\beta)s}\int_0^{u+rs} p(u+rs-v,T_\beta)\,\mathbb{P}(B\in \mathrm{d}v)\,\mathrm{d}s;$$

in the latter expression, the memoryless property of the exponential distribution has been exploited. We thus obtain the decomposition $\pi(\alpha,\beta) = \pi_1(\alpha,\beta) + \pi_2(\alpha,\beta)$, with

$$\pi_1(\alpha,\beta) := \int_0^\infty e^{-\alpha u}\int_0^\infty \lambda\, e^{-(\lambda+\beta)s}\int_{u+rs}^\infty \mathbb{P}(B\in \mathrm{d}v)\,\mathrm{d}s\,\mathrm{d}u,$$
$$\pi_2(\alpha,\beta) := \int_0^\infty e^{-\alpha u}\int_0^\infty \lambda\, e^{-(\lambda+\beta)s}\int_0^{u+rs} p(u+rs-v,T_\beta)\,\mathbb{P}(B\in \mathrm{d}v)\,\mathrm{d}s\,\mathrm{d}u.$$

We evaluate these two triple integrals separately. Interchanging the order of the integrals, we obtain that $\pi_1(\alpha,\beta)$ equals

$$\lambda\int_0^\infty\left(\int_0^v e^{-\alpha u}\left(\int_0^{(v-u)/r} e^{-(\lambda+\beta)s}\,\mathrm{d}s\right)\mathrm{d}u\right)\mathbb{P}(B\in \mathrm{d}v),$$

which, upon evaluating the inner integrals, reduces to

$$\frac{\lambda}{\lambda+\beta}\int_0^\infty\left(\frac{1-e^{-\alpha v}}{\alpha} - \frac{e^{-(\lambda+\beta)v/r}-e^{-\alpha v}}{\alpha-(\lambda+\beta)/r}\right)\mathbb{P}(B\in \mathrm{d}v). \tag{1.3}$$

This quantity can be interpreted in terms of the LST of B evaluated in specific values, to arrive at, with $s(\beta) := (\lambda+\beta)/r$,

$$\pi_1(\alpha,\beta) = \frac{\lambda}{\lambda+\beta}\left(\frac{1-\mathscr{B}(\alpha)}{\alpha} - \frac{\mathscr{B}(s(\beta))-\mathscr{B}(\alpha)}{\alpha-s(\beta)}\right).$$

Formulas of this form appear frequently throughout the book; see the further explanation of this fact in Appendix A.2.

We continue with evaluating the second integral, i.e., $\pi_2(\alpha,\beta)$. Performing the change of variable $w := u + rs$, we have that it equals

$$\frac{1}{r}\int_0^\infty e^{-\alpha u}\int_u^\infty \lambda\, e^{-s(\beta)(w-u)}\int_0^w p(w-v,T_\beta)\,\mathbb{P}(B\in \mathrm{d}v)\,\mathrm{d}w\,\mathrm{d}u.$$

As before, we swap the order of the integrals, so as to obtain

$$= \frac{}{A}\int_0^1 \int_E^1 4^{\,B^1V^o F}\, ?^1F \quad E\text{-})_V{}^o \int_0^F 4^{\,UD} 4^{B^1V^o D}\,\mathrm{d}D \;\; \mathrm{d}F \;\; \mathrm{P}^1 \;\; 2\,\mathrm{d}E^o$$

$$= \frac{}{A}\,\frac{1}{U \quad B^1V^o}\int_0^1 \int_E^1 4^{\,B^1V^o F} \quad 4^{\,UF}\; ?^1F \quad E\text{-})_V{}^o\,\mathrm{d}F \;\; \mathrm{P}^1 \;\; 2\,\mathrm{d}E^o\bullet$$

Now observe that

$$\int_E^1 4^{\,UF}\, ?^1F \quad E\text{-})_V{}^o\,\mathrm{d}F = 4^{\,UE}\int_0^1 4^{\,UF}\, ?^1F\text{-})_V{}^o\,\mathrm{d}F = 4^{\,UE} c^1U\text{-}\,V^o\text{-}$$

while a similar reasoning applies when replacing U by $B^1V^o\bullet$ Combining the above, we thus conclude that

$$c_2{}^1U\text{-}\,V^o = \frac{}{A}\,\frac{1}{B^1V^o \quad U}\;\; 1^1U^o c^1U\text{-}\,V^o \quad 1^1B^1V^{oo} c^1B^1V^o\text{-}\,V^o \;\;;$$

for $U = B^1V^o$ this expression is defined in the obvious manner. Upon adding up the expressions that we found for $c_1{}^1U\text{-}\,V^o$ and $c_2{}^1U\text{-}\,V^o$, observing that $c_2{}^1U\text{-}\,V^o$ contains a term involving $c^1U\text{-}\,V^o$, we thus end up with an expression for $c^1U\text{-}\,V^o$ in terms of the same quantity. After some elementary algebraic manipulations, this leads to

$$c^1U\text{-}\,V^o = A\frac{}{\_\;{}_\text{¸}\;V}\,\frac{B^1V^o \quad U}{A^1B^1V^o \quad U^o \quad \_1^1U^o}\,\frac{1 \quad 1^1U^o}{U}$$

$$A\frac{}{\_\;{}_\text{¸}\;V}\,\frac{1^1U^o \quad 1^1B^1V^{oo}}{A^1B^1V^o \quad U^o \quad \_1^1U^o} \quad \frac{\_\,1^1B^1V^{oo}\, c^1B^1V^o\text{-}\,V^o}{A^1B^1V^o \quad U^o \quad \_1^1U^o}\bullet$$

Observe that the right-hand side contains the unknown quantity $c^1B^1V^o\text{-}\,V^o$, but this can be identified by using the property that a root of the denominator is necessarily also a root of the numerator (where it should be realized that $c^1U\text{-}\,V^o$ is finite for all $U > 0$ and $V \;¡\; 0$). It is easily seen that the equation $A^1B^1V^o \quad U^o \quad \_1^1U^o = 0$, or equivalently the equation

$$1^1U^o = \;{}^1U^o := 1 \;{}_\text{¸}\; \frac{V}{\_} \quad \frac{AU}{\_}\text{-}$$

has for any $V \;¡\; 0$ a unique positive root, say k^1V^o; to this end, observe that $1^{1\;o}$ is convex with $1^10^o = 1$ and $1^1\,1^{\,o} = 0$, whereas ${}^{1\;o}$ is linear with ${}^10^o \;¡\; 1$ and ${}^1\,1^{\,o} = \;\; 1\bullet$ We end up with

$$c^1B^1V^o\text{-}\,V^o = \frac{A}{\_\;{}_\text{¸}\;V}\quad \frac{B^1V^o \quad k^1V^o}{1^1B^1V^{oo}}\,\frac{1 \quad 1^1k^1V^{oo}}{k^1V^o} \quad \frac{1^1k^1V^{oo} \quad 1^1B^1V^{oo}}{1^1B^1V^{oo}}$$

$$= \frac{A}{\_\;{}_\text{¸}\;V}\quad \frac{B^1V^{o1}1 \quad 1^1k^1V^{ooo} \quad k^1V^{o1}1 \quad 1^1B^1V^{ooo}}{1^1B^1V^{oo}\,k^1V^o}\;\;\bullet$$

Now define

$$i^1U^o := \log \mathrm{E}\,4^{\,U.\;{}^11^o} = AU \quad \_^11 \quad 1^1U^{oo}\bullet$$

In the literature on Lévy processes, the function $\varphi(\cdot)$ is commonly referred to as the *Laplace exponent* of the compound Poisson process defined by the arrival rate λ, the premium rate r, and the distribution of the i.i.d. claims B_i. It is readily verified that the function $\psi(\cdot)$, as defined above, is the inverse of the Laplace exponent $\varphi(\cdot)$ — in case $\varphi'(0) < 0$ actually the *right* inverse. Plugging the expression for $c(\psi(q), q)$ into $c(\alpha, q)$, after considerable calculus one arrives at the compact relation

$$\begin{aligned} c(\alpha,q) &= \frac{\lambda}{\varphi(\alpha)-q}\left(\frac{1-b(\psi(q))}{\psi(q)}-\frac{1-b(\alpha)}{\alpha}\right)\\ &= \frac{1}{\varphi(\alpha)-q}\left(\frac{\varphi(\alpha)-r\alpha}{\alpha}-\frac{q-r\psi(q)}{\psi(q)}\right)\\ &= \frac{1}{\varphi(\alpha)-q}\left(\frac{\varphi(\alpha)}{\alpha}-\frac{q}{\psi(q)}\right). \end{aligned} \tag{1.4}$$

See Appendix A.3 for a proof of the fact that $\alpha = \psi(q)$ is the only zero of $\varphi(\alpha) - q$ in the right-half α-plane. From the above expression, using an Abelian theorem, we can derive the transform pertaining to the all-time ruin probability by letting q go to 0. We impose the net-profit condition, as was introduced in Remark 1.1, to rule out that $p(u) = 1$ for any u. Under this condition we have that $\varphi'(0) > 0$, so that $\varphi(\alpha)$ is increasing on $[0, \infty)$. As a consequence, $\psi(q)$ is the inverse of $\varphi(\alpha)$ on $[0, \infty)$ with $\varphi(0) = \psi(0) = 0$; see Figure 1.3. This implying that $1/\psi'(0) = \varphi'(0)$, we thus obtain

$$c(\alpha) = \frac{1}{\varphi(\alpha)}\left(\frac{\varphi(\alpha)}{\alpha}-\frac{1}{\psi'(0)}\right) = \frac{1}{\alpha}-\frac{\varphi'(0)}{\varphi(\alpha)}.$$

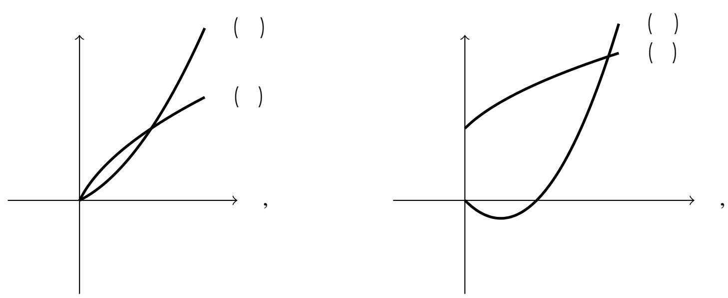

Fig. 1.3 The functions $\varphi(\alpha)$ and $\psi(q)$ with $\varphi'(0) > 0$ (left panel) and with $\varphi'(0) < 0$ (right panel). In the former case $\psi(0) = 0$, whereas in the latter case $\psi(0) > 0$.

We now use the results obtained above to derive expressions for the transforms of $\bar{p}(T_q)$ and $\bar{p}(\infty)$. Using the relation (1.1) that translates $r(\alpha, q)$ in terms of $c(\alpha, q)$, we eventually find the following result. It is sometimes referred to as the time-dependent version of the *Pollaczek-Khinchine formula*.

Theorem 1.1 *For any $\alpha > 0$ and $q \geq 0$,*

$$\pi(\alpha,q) = \frac{\alpha - \Phi(q)}{\phi(\alpha) - q}\,\frac{q}{\Phi(q)}.$$

We also obtain the following result by letting $q \downarrow 0$, providing the transform of $\bar{X}(\infty)$, known as the (classical version of the) Pollaczek-Khinchine formula. We need to impose the net-profit condition, introduced in Remark 1.1, to make sure that $\bar{X}(\infty)$ is finite. This formula is also one of the most prominent results in the M/G/1 context; see also Theorem B.2.

Corollary 1.1 *For any $\alpha > 0$, under the net-profit condition,*

$$\pi(\alpha) := \mathbb{E}\,e^{-\alpha\bar{X}(\infty)} = \pi(\alpha,0) = \frac{\alpha\,\phi'(0)}{\phi(\alpha)}. \tag{1.5}$$

We refer to Exercise 1.1 for a procedure that uses Theorem 1.1 to recursively evaluate all moments of the running maximum $\bar{X}(T_q)$.

▷ *Geometric sum representation.* The formula (1.5) is one of the versions of the classical Pollaczek-Khinchine formula. It can be alternatively written as

$$\pi(\alpha) = \frac{\alpha(r - \lambda\mathbb{E}B)}{\alpha r - \lambda(1 - b(\alpha))} = \left(1 - \frac{\lambda\mathbb{E}B}{r}\right)\Big/\left(1 - \frac{\lambda}{r}\,\frac{1 - b(\alpha)}{\alpha}\right).$$

Observe that $(1 - b(\alpha))/(\alpha\mathbb{E}B)$ is the LST of a non-negative random variable $\bar{B}$ with density $f_{\bar{B}}(x) := \mathbb{P}(B > x)/\mathbb{E}B$; indeed, the LST of $\bar{B}$ reads (cf. Appendix A.3):

$$\mathbb{E}\,e^{-\alpha\bar{B}} = \int_0^\infty e^{-\alpha x}\,\frac{\mathbb{P}(B > x)}{\mathbb{E}B}\,\mathrm{d}x = \frac{1 - b(\alpha)}{\alpha\mathbb{E}B}.$$

The random variable $\bar{B}$ is called the 'residual' of B, owing to an interpretation from renewal theory. We conclude that $|(1 - b(\alpha))/(\alpha\mathbb{E}B)| \leq 1$ for $\alpha > 0$. Hence we have the following representation of $\pi(\alpha)$ in terms of a convergent sum:

$$\pi(\alpha) = \left(1 - \frac{\lambda\mathbb{E}B}{r}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda\mathbb{E}B}{r}\right)^n\left(\frac{1 - b(\alpha)}{\alpha\mathbb{E}B}\right)^n.$$

Define $p := 1 - \lambda\mathbb{E}B/r \in (0,1)$, and let N be geometrically distributed with success probability p, meaning that, for $n = 0,1,\ldots$,

$$\mathbb{P}(N = n) = (1-p)^n p.$$

In addition, let $\bar{B}^{*i}$ be a random variable defined as the sum of i i.i.d. copies of $\bar{B}$. Then the above leads to an appealing representation of the all-time maximum, in self-evident notation.

Proposition 1.1 *The following distributional equality applies: under the net-profit condition,*

$$\bar{.}\,{}^{1}1^{\circ} \stackrel{\mathrm{d}}{=} \tilde{O}_{8=1}\ \bar{}_{8} = \bar{}^{\,\text{¢}}\ \text{-}$$

an empty sum being defined as zero.

Informally, one says that the all-time running maximum is distributed as a geometric number of i.i.d. samples from the residual claim size distribution.

B *Wiener-Hopf decomposition.* We conclude this section with a number of observations concerning the running minimum $\underline{.}\,{}^{1}C^{\circ}$, leading to the celebrated Wiener-Hopf decomposition. In the first place, we provide an elementary argument that reveals the distribution of $\underline{.}\,{}^{1})_{V}{}^{\circ}$, i.e., minus the running minimum at an exponentially distributed epoch.

Lemma 1.1 *For any V ¡ 0, $\underline{.}\,{}^{1})_{V}{}^{\circ}$ has an exponential distribution with mean* $1\bullet k^{1}V^{\circ}$.

Proof. It is readily verified that the process ${}^{1}C^{\circ} := 4^{\ i\,{}^{1}U^{\circ}C}\,4^{\ U.\ {}^{1}C^{\circ}}$ is a mean-1 martingale. Define $f^{\,1}E^{\circ}$ as the first time $.\ {}^{1}C^{\circ}$ crosses the level E, for some given E ¡ 0. Now consider, with ‘optional sampling’ [13], the martingale ${}^{1}C^{\circ}$ at the stopping time $f^{\,1}E^{\circ}$. Because $.\ {}^{1}C^{\circ}$ does not have jumps in the downward direction, it follows that $.\ {}^{1}f^{\,1}E^{\circ\circ} = E$; cf. Figure 1.2. This observation leads to the identity

$$1 = \mathrm{E}\ \ {}^{1}0^{\circ} = \mathrm{E}\ \ {}^{1}f^{\,1}E^{\circ\circ} = \mathrm{E}\ \ 4^{\ i\,{}^{1}U^{\circ}\,f^{\,1}E^{\circ}}1\text{f}\,f^{\,1}E^{\circ}\ \ddot{Y}\ 1\text{g}\ \ 4^{UE}\bullet$$

(Note that, formally, the application of ‘optional sampling’ requires a specific condition to be met; see e.g. [4, Corollary III.1.4] for an argument that applies in our specific case.) Plugging in $U = k^{1}V^{\circ}$, we thus obtain an expression for the Laplace transform of the first passage time $f^{\,1}E^{\circ}$:

$$\mathrm{E}\ \ 4^{\ V\,f^{\,1}E^{\circ}}1\text{f}\,f^{\,1}E^{\circ}\ \ddot{Y}\ 1\text{g}\ = 4^{\ k^{1}V^{\circ}E}\bullet \tag{1.6}$$

Then the stated exponentiality immediately follows from the (obvious) equivalence of the events f $\underline{.}\,{}^{1})_{V}{}^{\circ} > E$g and ff $^{1}E^{\circ}$ 6 $)_{V}$g: recalling Equation (1.2),

$$\mathrm{P}^{1}\ \underline{.}\,{}^{1})_{V}{}^{\circ} > E^{\circ} = \mathrm{P}^{1}f^{\,1}E^{\circ}\ 6\ )_{V}{}^{\circ} = \mathrm{E}\ \ 4^{\ V\,f^{\,1}E^{\circ}}1\text{f}\,f^{\,1}E^{\circ}\ \ddot{Y}\ 1\text{g}\ = 4^{\ k^{1}V^{\circ}E}\bullet$$

This proves the claim.

With this lemma at our disposal, a remarkable independence property can be derived. To this end we observe that the following claims apply.

First note that

$$\mathrm{E}\,4^{\ U.\ {}^{1})_{V}{}^{\circ}} = \int_{0}^{1} V4^{\ VC}4^{\ i\,{}^{1}U^{\circ}C}\,\mathrm{d}C = \frac{V}{V\ \ i^{\,1}U^{\circ}}\bullet$$

Secondly, by an elementary time-reversal argument, it can be seen that $\bar{.}\,{}^{1})_{V}{}^{\circ}$ $.\ {}^{1})_{V}{}^{\circ}$ is distributed as $\underline{.}\,{}^{1})_{V}{}^{\circ}$. This implies, by Lemma 1.1, that $\bar{.}\,{}^{1})_{V}{}^{\circ}$ $.\ {}^{1})_{V}{}^{\circ}$

has an exponential distribution with mean $1/\psi(\vartheta)$, so that

$$\mathbb{E}\, e^{-\alpha(\bar{X}(T_\vartheta)-X(T_\vartheta))} = \frac{\psi(\vartheta)}{\psi(\vartheta)+\alpha}.$$

By Theorem 1.1 and the above results, we obtain that

$$\begin{aligned}\mathbb{E}\, e^{-\alpha\bar{X}(T_\vartheta)}\,\mathbb{E}\, e^{-\alpha(X(T_\vartheta)-\bar{X}(T_\vartheta))} &= \frac{\alpha-\psi(\vartheta)}{\varphi(\alpha)-\vartheta}\,\frac{\vartheta}{\psi(\vartheta)}\;\frac{\psi(\vartheta)}{\psi(\vartheta)-\alpha}\\ &= \frac{\vartheta}{\vartheta-\varphi(\alpha)} = \mathbb{E}\, e^{-\alpha X(T_\vartheta)}.\end{aligned}$$

As evidently $X(T_\vartheta) = \bar{X}(T_\vartheta) + (X(T_\vartheta)-\bar{X}(T_\vartheta))$, combining the above observations leads to the following result.

Proposition 1.2 *The random variables $\bar{X}(T_\vartheta)$ and $\bar{X}(T_\vartheta)-X(T_\vartheta)$ are independent. The former has a Laplace-Stieltjes transform that is given by Theorem 1.1, whereas the latter has the same distribution as $-\underline{X}(T_\vartheta)$, i.e., has an exponential distribution with mean $1/\psi(\vartheta)$.*

The above factorization is often referred to as the *Wiener-Hopf decomposition.* It can be intuitively understood as follows. At the moment of reaching the running maximum $\bar{X}(T_\vartheta)$, the position of this running maximum does not provide any information on the distance by which the process goes down until the killing epoch (i.e., $\bar{X}(T_\vartheta)-X(T_\vartheta)$). In this reasoning the memoryless property of the exponentially distributed killing epoch plays a crucial role, next to the independent increments property of the $X(t)$ process; for any deterministic $t > 0$ the random quantities $\bar{X}(t)$ and $\bar{X}(t)-X(t)$ are obviously *not* independent.

Remark 1.4 As mentioned, in Exercise 1.1 we develop a procedure that recursively evaluates all moments of the running maximum $\bar{X}(T_\vartheta)$ exploiting the time-dependent version of the Pollaczek-Khinchine formula, i.e., Theorem 1.1. Observe that the Wiener-Hopf decomposition provides an efficient way to compute the first moment:

$$\mathbb{E}\,\bar{X}(T_\vartheta) = \mathbb{E}\,X(T_\vartheta) - \mathbb{E}\,\underline{X}(T_\vartheta).$$

Due to Proposition 1.2, $\mathbb{E}\,\underline{X}(T_\vartheta) = -1/\psi(\vartheta)$. Also, $\mathbb{E}\,X(T_\vartheta) = \mathbb{E}\,X(1)/\vartheta = -\varphi'(0)/\vartheta.$ We conclude that

$$\mathbb{E}\,\bar{X}(T_\vartheta) = -\frac{\varphi'(0)}{\vartheta} + \frac{1}{\psi(\vartheta)}.$$

As can be verified, under the net-profit condition, we can find the mean of the all-time supremum by letting $\vartheta \downarrow 0$. Indeed,

$$\begin{aligned}\mathbb{E}\,\bar{X}(\infty) &= \lim_{\vartheta\downarrow 0}\frac{\vartheta-\varphi'(0)\psi(\vartheta)}{\vartheta\,\psi(\vartheta)} = \lim_{\vartheta\downarrow 0}\frac{1-\varphi'(0)\psi'(\vartheta)}{\vartheta\,\psi'(\vartheta)+\psi(\vartheta)}\\ &= \lim_{\vartheta\downarrow 0}\frac{-\varphi'(0)\psi''(\vartheta)}{\vartheta\psi''(\vartheta)+2\psi'(\vartheta)} = -\frac{\varphi'(0)\psi''(0)}{2\psi'(0)}.\end{aligned}$$

applying L'Hopital's rule twice. Differentiating $k(\varphi(U)) = U$ twice, we obtain that $k''(0) = -\varphi''(0)/\varphi'(0)^3$. We conclude that

$$\mathbb{E}\,\bar{\tau}(1) = \frac{\varphi''(0)}{2\,\varphi'(0)^2} = \frac{\lambda\,\mathbb{E}B^2}{2(r-\lambda\,\mathbb{E}B)^2}.$$

In a similar way, from the identity $\mathrm{Var}\,\tau(1) = \mathrm{Var}\,\bar{\tau}(1) + \mathrm{Var}\,\underline{\tau}(1)$, an expression for $\mathrm{Var}\,\bar{\tau}(1)$ can be derived. ^

1.4 Method 2: ladder heights, busy periods

In this section we re-establish our expression for $r(u, v)$, relying on the concept of *ladder heights* of the process $Y(t)$, and the concept of *busy periods* of the associated queueing process $Q(t)$. We refer to Appendix B.1 for a discussion of such busy periods.

B *Definitions.* We start by introducing the concept of ladder heights. Define $g_0 = 0$, and for $i = 1, 2, \ldots$,

$$g_i := \inf\Big\{t \geq 0 : Y\Big(t + \sum_{j=1}^{i-1} g_j\Big) > Y\Big(\sum_{j=1}^{i-1} g_j\Big)\Big\},\quad H_i := Y\Big(\sum_{j=1}^{i} g_j\Big) - Y\Big(\sum_{j=1}^{i-1} g_j\Big).$$

The i-th ladder height H_i can be interpreted as the difference between the process' i-th and $(i-1)$-st record value, while g_i represents the time elapsed between the epochs at which these two record values are attained; see Figure 1.4 for a pictorial illustration. It is clear that $(H_i, g_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random vectors; let (H, g) be the corresponding generic random vector. A first observation is that $\bar{\tau}(u)$ is distributed as the sum of a geometric number (with a specific success probability, yet to be determined) of i.i.d. copies of H. This property we will use to re-establish our expression for $r(u, v)$.

The second concept we will extensively work with is that of busy periods. These are uninterrupted intervals in which the associated queueing process is positive. It is directly seen that in our setup these busy periods are i.i.d., say distributed as a generic random variable f. With B sampled independently of the process $Y(t)$, the busy period f is distributed as the first time $Y(t)$ crosses the (stochastic) level B. Note that f can be defective if the net-profit condition (see Remark 1.1) is not fulfilled. With $f(x)$ as defined in Section 1.3, we thus get

$$\mathbb{E}\,e^{-\alpha f}\,1\{f < \infty\} = \int_0^\infty \mathbb{E}\,e^{-\alpha f(x)}\,1\{f(x) < \infty\}\,\mathbb{P}(B \in dx) = \int_0^\infty e^{-k(\alpha)x}\,\mathbb{P}(B \in dx) = b(k(\alpha)).$$

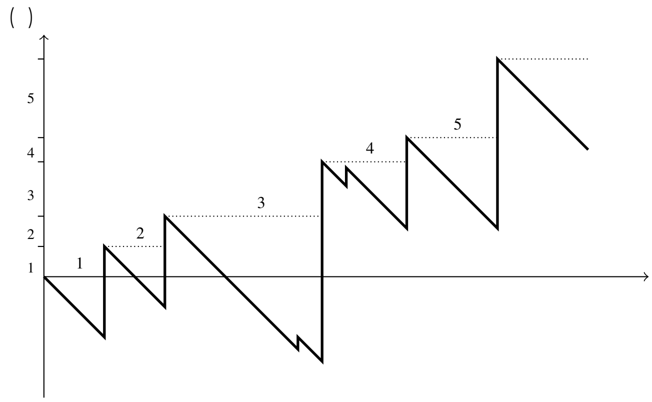

Fig. 1.4 Net cumulative claim process . ¹C°, the ladder height process ¹ =°=, and the 'inter-ladder times' ¹g=°=. The dotted process is .̄ ¹C°•

relying on Equation (1.6). Using the definition of *i* ¹ °, we find that

$$V = i^1 k^1 V^{oo} = Ak^1 V^o \quad \_^1 1 \quad 1^1 k^1 V^{ooo} \bullet$$

Hence, our expression for E ¹4 $^{V f}$ 1f *f* Ÿ 1g° can be rewritten as follows; cf. also Theorem B.3 and in particular Equation (B.6).

Lemma 1.2 *For any V ¡* 0*,*

$$\mathrm{E}\ 4^{\ V f}\ 1 f\ f\ \ddot{Y}\ 1 g\ = \frac{V\ \text{¸}\ \_}{\_}\quad \frac{A}{\_} k^1 V^o \bullet$$

B *An auxiliary result involving busy periods.* For purposes that will become clear later, we now focus on the evaluation of the object

$$b^1 U\text{-} V^o := \mathrm{E}\ 4^{\ U.\ ^1)V^o}\ 1 f.\ ^1)_V{}^o = \underline{.}\ ^1)_V{}^o g\ = \mathrm{E}\ 4^{\ U\underline{.}\ ^1)V^o}\ 1 f.\ ^1)_V{}^o = \underline{.}\ ^1)_V{}^o g\ \bullet$$

Proposition 1.3 *For any U >* 0 *and V ¡* 0*,*

$$b^1 U\text{-} V^o = \frac{V}{A k^1 V^o \quad AU}\bullet$$

Proof. Recall that _.¹C°•*A* can be interpreted as the associated queue's idle time in »0-C¼, which in this proof we denote by *!* ¹C°. As a consequence, we can equivalently look at the object

$$b^1 U\text{-} V^o = \mathrm{E}\ 4^{AU!\ ^1)V^o}\ 1 f \&\ ^1)_V{}^o = 0 g\ \bullet$$

Conditioning on the time of the first event, which can be either the killing time or the start of a busy period, we find by exploiting the underlying regenerative structure

$$\kappa(\alpha,\beta)=\frac{\beta}{\lambda+\beta-r\alpha}+\frac{\lambda}{\lambda+\beta-r\alpha}\,\mathbb{P}(\tau\leq T_\beta)\,\kappa(\alpha,\beta).$$

Recalling that $\mathbb{P}(\tau\leq T_\beta)$ can be rewritten as $\mathbb{E}(e^{-\beta\tau}1\{\tau<\infty\})$, we find after a few elementary algebraic steps, using the expression for $\mathbb{E}(e^{-\beta\tau}1\{\tau<\infty\})$ that we identified in Lemma 1.2,

$$\kappa(\alpha,\beta)=\frac{\beta}{\lambda(1-\mathbb{E}(e^{-\beta\tau}1\{\tau<\infty\}))+\beta-r\alpha}=\frac{\beta}{r\psi(\beta)-r\alpha}$$

as desired.

This result has a few interesting ramifications. In the first place it provides us with an explicit formula for the probability that the queue is empty after an exponentially distributed time. Indeed, with $\pi_{T_\beta}:=\mathbb{P}(Q(T_\beta)=0)=\mathbb{P}(X(T_\beta)=\underline{X}(T_\beta))$, we find that

$$\pi_{T_\beta}=\kappa(0,\beta)=\frac{\beta}{r\psi(\beta)}=\frac{\beta}{\beta+\lambda(1-\mathbb{E}(e^{-\beta\tau}1\{\tau<\infty\}))}.$$

In the second place, recalling that we know by Lemma 1.1 that $-\underline{X}(T_\beta)$ has an exponential distribution with mean $1/\psi(\beta)$, and slightly rewriting $\kappa(\alpha,\beta)$, we conclude that

$$\kappa(\alpha,\beta)=\frac{\psi(\beta)}{\psi(\beta)-\alpha}\frac{\beta}{r\psi(\beta)}=\mathbb{E}\,e^{\alpha\underline{X}(T_\beta)}\,\mathbb{P}(X(T_\beta)=\underline{X}(T_\beta)).$$

From this equality we observe that the quantities $\underline{X}(T_\beta)$ and $1\{X(T_\beta)=\underline{X}(T_\beta)\}$, appearing in the definition of $\kappa(\alpha,\beta)$, are actually independent.

∘ *Characterization of the ladder height distribution.* Our following objective is to compute, relying on the expression for $\kappa(\alpha,\beta)$ that we derived above,

$$\chi(\alpha,\beta):=\mathbb{E}\left(e^{-\alpha X(g)-\beta g}1\{g<\infty\}\right),$$

i.e., the joint transform of the ladder height and the time elapsed since the previous record value, intersected with the event that there is a new record value.

Proposition 1.4 *For any $\alpha>0$ and $\beta\geq 0$,*

$$\chi(\alpha,\beta)=1-\frac{\beta-\varphi(\alpha)}{r\psi(\beta)-r\alpha}.$$

Proof. The starting point is the decomposition

$$\frac{\beta}{\beta-\varphi(\alpha)}=\mathbb{E}\,e^{\alpha X(T_\beta)}=\chi_1(\alpha,\beta)+\chi_2(\alpha,\beta),$$

where

$$\chi_1(\alpha,\beta):=\mathbb{E}\left(e^{\alpha X(T_\beta)}1\{g\geq T_\beta,\,g<\infty\}\right),$$
$$\chi_2(\alpha,\beta):=\mathbb{E}\left(e^{\alpha X(T_\beta)}1\{g\leq T_\beta,\,g<\infty\}\right).$$

We continue by analyzing $[_1{}^1U\text{-}\,V^{o}$. Recalling the definition of $b^1U\text{-}\,V^{o}$, it is directly seen that

$$[_1{}^1U\text{-}\,V^{o} = \mathrm{E}\; 4^{\;U.\,{}^1)_V{}^{o}}\, 1\mathrm{f}\bar{.}{}^1)_V{}^{o} = 0\mathrm{g}$$
$$= \mathrm{E}\; 4^{\;U.\,{}^1)_V{}^{o}}\, 1\mathrm{f}.\,{}^1)_V{}^{o} \quad \underline{.}{}^1)_V{}^{o} = 0\mathrm{g} \;=\; b^1U\text{-}\,V^{o}\text{-}$$

which we know from Proposition 1.3. In addition, we express $[_2{}^1U\text{-}\,V^{o}$ as a triple integral:

$$[_2{}^1U\text{-}\,V^{o} = \int_{C=0}^{1} V4^{\;VC} \int_{B=0}^{C}\int_{H=0}^{1} 4^{\;UH}\, \mathrm{E}\; 4^{\;U^1.\,{}^1C^{o}\;\;.\,{}^1g^{oo}}\,\mathrm{j}\quad = H\text{-}\,g = B$$
$$\mathrm{P}^1 \quad 2\,\mathrm{d}H\text{-}\,g\; 2\;\mathrm{d}B^{o}\,\mathrm{d}C$$
$$= \int_{C=0}^{1} V4^{\;VC} \int_{B=0}^{C}\int_{H=0}^{1} 4^{\;UH} 4^{\,i\,{}^1U^{o1}C\;\;B^{o}}\,\mathrm{P}^1 \quad 2\,\mathrm{d}H\text{-}\,g\; 2\;\mathrm{d}B^{o}\,\mathrm{d}C\bullet$$

In order to evaluate $[_2{}^1U\text{-}\,V^{o}$, the next step is to swap the order of the integrals, so that we can first perform the (easy) integration over C:

$$\int_{B=0}^{1}\int_{H=0}^{1}\int_{C=B}^{1} V4^{\;VC} 4^{\,i\,{}^1U^{o}C}\,\mathrm{d}C\;\; 4^{\;UH}\, 4^{\;\;i\,{}^1U^{o}B}\,\mathrm{P}^1 \quad 2\,\mathrm{d}H\text{-}\,g\; 2\;\mathrm{d}B^{o}$$
$$= \frac{V}{V \quad i\,{}^1U^{o}} \int_{B=0}^{1}\int_{H=0}^{1} 4^{\;UH} 4^{\;VB}\,\mathrm{P}^1 \quad 2\,\mathrm{d}H\text{-}\,g\; 2\;\mathrm{d}B^{o} = \frac{V}{V \quad i\,{}^1U^{o}}\,[^1U\text{-}\,V^{o}\bullet$$

Combining the above, after some algebra we find the desired expression.

B *Transform of the running maximum.* Using the geometric-sum representation mentioned at the beginning of this section, we can now compute the transform $d^1U\text{-}\,V^{o}$ of the running maximum $\bar{.}{}^1)_V{}^{o}$. Indeed, we recover Theorem 1.1, as follows. First observe that the number of record values has a geometric distribution with success probability $\mathrm{P}^1g\;\;j\;\;)_V{}^{o} = 1 \quad [^10\text{-}\,V^{o}$. Hence we can write

$$\mathrm{E}4^{\;U.\,{}^1)_V{}^{o}} = \mathrm{E}4^{\;U\;\acute{\imath}_{8=1}\;\;8} = \tilde{\text{①}}_{:=0}\; [^10\text{-}\,V^{o\;\;:}\;\; 1 \quad [^10\text{-}\,V^{o} \quad \mathrm{E}^1 4^{\;U} \quad \mathrm{j}\,g\;\ddot{Y}\;)_V{}^{o\;\;:}\;\bullet$$

Noting that, cf. Remark 1.3,

$$[^1U\text{-}\,V^{o} = \mathrm{E}\;\;4^{\;U \quad Vg}\,1\mathrm{f}g\;\ddot{Y}\;1\mathrm{g} \;=\; \mathrm{E}\;\;4^{\;U}\;\;1\mathrm{f}g\;\ddot{Y}\;)_V\mathrm{g}\;\text{-}$$

it now follows, by combining the above and relying on the expression for $[^1U\text{-}\,V^{o}$ that we have identified in Proposition 1.4,

$$r^1U\text{-}\,V^{o} = \tilde{\text{①}}_{:=0}\;\; [^1U\text{-}\,V^{o\;\;:}\;\; 1 \quad [^10\text{-}\,V^{o}$$

$$= \sum_{i=0}^{\infty}\left(1-\frac{\beta-\varphi(\alpha)}{r\psi(\beta)-r\alpha}\right)^{i}\frac{\beta}{r\psi(\beta)} = \frac{\alpha-\psi(\beta)}{\varphi(\alpha)-\beta}\,\frac{\beta}{\psi(\beta)},$$

as desired.

1.5 Method 3: Kella-Whitt martingale

Where in Section 1.3 we identified the transform $\kappa(\alpha,\beta)$ of $\bar{Y}(T_\beta)$ by conditioning on the first event (either a claim arrival or killing), in Section 1.4 we developed an alternative method in which we combined the representation of the running maximum $\bar{Y}(T_\beta)$ and the representation in terms of the M/G/1 workload at an exponential time $Q(T_\beta)$. In this section we discuss a third approach to find $\kappa(\alpha,\beta)$, again relying on the duality presented in Section 1.2, but now working with a martingale involving the queueing process $Q(t)$.

▷ *Definition and martingale property.* Our objective is to identify the LST of $Q(T_\beta)$, which, due to the duality, coincides with the LST of $\bar{Y}(T_\beta)$. To this end, we define the process

$$M(t) := \varphi(\alpha)\int_0^t e^{-\alpha Q(s)}\,\mathrm{d}s + 1 - e^{-\alpha Q(t)} + \alpha\underline{Y}(t).$$

Lemma 1.3 *The process $M(t)$ is a martingale with respect to $\mathcal{F}(t)$, i.e., the natural filtration pertaining to $\{Y(s): s\in[0,t]\}$.*

We proceed by providing an informal (but intuitively appealing) backing of this claim. More concretely, our objective is to verify that, as $\delta\downarrow 0$, up to $o(\delta)$-terms, $\mathbb{E}(M(t+\delta)\,|\,\mathcal{F}(t)) = M(t)$.

First consider the situation that the path $\{Y(s): s\in[0,t]\}$ is such that, for some $\delta > 0$ (which can be made arbitrarily small), $Q(t) > r\delta$. Then $\underline{Y}(t) = \underline{Y}(t+\delta)$ and $Q(s) = Q(t) + Y(s) - Y(t)$ for $s\in[t,t+\delta]$ (as a consequence of the fact that the queue will not empty between t and $t+\delta$; recall the interpretation of the process $\underline{Y}(t)$ in terms of the queue's cumulative idle time). Supposing we have access to $\mathcal{F}(t)$ (and, for conciseness, throughout the derivation below suppressing that we condition on this $\mathcal{F}(t)$),

$$\begin{aligned}\mathbb{E}\,M(t+\delta) &= \varphi(\alpha)\left(\int_0^t e^{-\alpha Q(s)}\,\mathrm{d}s + e^{-\alpha Q(t)}\,\mathbb{E}\int_0^\delta e^{-\alpha Y(s)}\,\mathrm{d}s\right)\\ &\quad + 1 - e^{-\alpha Q(t)}\,\mathbb{E}e^{-\alpha Y(\delta)} + \alpha\underline{Y}(t) = M(t),\end{aligned}$$

as desired; here we have used that $\mathbb{E}e^{-\alpha Y(\delta)} = e^{\varphi(\alpha)\delta}$ and

$$\int_0^\delta \mathbb{E}\,e^{-\alpha Y(s)}\,\mathrm{d}s = \int_0^\delta e^{\varphi(\alpha)s}\,\mathrm{d}s = \frac{e^{\varphi(\alpha)\delta}-1}{\varphi(\alpha)}.$$

Now suppose that $\{Y(s) : s \in [0,t]\}$ is such that $Q(t) = 0$. Observe that, up to $o(\Delta)$-terms, $Q(t+\Delta)$ is distributed as B with probability $\lambda\Delta$, and is 0 otherwise. In addition $\underline{Y}(t) = Y(t)$, as the process and its running minimum process coincide during the queue's idle times. Hence, again ignoring $o(\Delta)$-terms, we obtain

$$\varphi(\alpha)\mathbb{E}\int_0^{t+\Delta} e^{-\alpha Q(s)}\,\mathrm{d}s = \varphi(\alpha)\int_0^{t} e^{-\alpha Q(s)}\,\mathrm{d}s + \varphi(\alpha)\Delta,$$

and, along the same lines,

$$\begin{aligned}\mathbb{E}\,e^{-\alpha Q(t+\Delta)} &= e^{-\alpha Q(t)}\big(1-\lambda\Delta+\lambda\Delta b(\alpha)\big)\\ &= e^{-\alpha Q(t)} - \lambda\Delta + \lambda\Delta b(\alpha),\end{aligned}$$

and

$$\alpha\mathbb{E}\,\underline{Y}(t+\Delta) = \alpha\underline{Y}(t) - \alpha\Delta.$$

From the above, we find, up to $o(\Delta)$-terms,

$$\mathbb{E}\,M(t+\Delta) = M(t) + \big(\varphi(\alpha) + \lambda(1-b(\alpha)) - \alpha\big)\Delta = M(t),$$

as desired. Combining the findings corresponding to $Q(t) > 0$ and $Q(t) = 0$, we complete our informal reasoning as to why the process $M(t)$ is a martingale. In the literature $M(t)$ is commonly referred to as the *Kella-Whitt martingale*.

▷ *Transform of workload at an exponential epoch.* Now that we know that $M(t)$ is a martingale, we demonstrate its use by providing a highly efficient derivation of the transform of $Q(T_\nu)$. Using 'optional sampling' with the stopping time T_ν and recalling that $Q(0) = 0$, we have that $0 = \mathbb{E}\,M(0) = \mathbb{E}\,M(T_\nu)$. In other words,

$$0 = \varphi(\alpha)\,\mathbb{E}\int_0^{T_\nu} e^{-\alpha Q(s)}\,\mathrm{d}s + 1 - \mathbb{E}\,e^{-\alpha Q(T_\nu)} + \alpha\mathbb{E}\,\underline{Y}(T_\nu). \tag{1.7}$$

(A sufficient condition for the use of 'optional sampling' is provided in [4, Theorem II.4.8]: the equality (1.7) holds if $\mathbb{E}\,\sup_{t\in[0,T_\nu]}|M(t)| < \infty$. In our case this finiteness directly follows from the upper bound

$$|M(t)| \leqslant |\varphi(\alpha)|T_\nu + 2 + \alpha T_\nu,$$

for any $t \in [0, T_\nu]$.)

Then observe that, swapping the order of integration,

$$\begin{aligned}\mathbb{E}\int_0^{T_\nu} e^{-\alpha Q(s)}\,\mathrm{d}s &= \int_0^\infty \nu e^{-\nu t}\int_0^t \mathbb{E}\,e^{-\alpha Q(s)}\,\mathrm{d}s\,\mathrm{d}t\\ &= \int_0^\infty\left(\int_s^\infty \nu e^{-\nu t}\mathrm{d}t\right)\mathbb{E}\,e^{-\alpha Q(s)}\,\mathrm{d}s = \int_0^\infty e^{-\nu s}\,\mathbb{E}\,e^{-\alpha Q(s)}\,\mathrm{d}s\end{aligned}$$

$$= \frac{1}{\beta}\,\mathbb{E}\,e^{-\alpha Q(T_\beta)}.$$

Now solving $\mathbb{E}\,e^{-\alpha Q(T_\beta)}$ from (1.7), we readily obtain

$$\mathbb{E}\,e^{-\alpha Q(T_\beta)} = \frac{\beta}{\varphi(\alpha)-\beta}\left(-\alpha\,\mathbb{E}\,\underline{X}(T_\beta) - 1\right).$$

We are left to identify the unknown quantity $\mathbb{E}\,\underline{X}(T_\beta)$. To this end, note that (fixing $\beta > 0$) any root $\alpha > 0$ of the denominator should be root of the numerator as well, so as to render a number between 0 and 1. As a consequence, using that $\alpha = \Psi(\beta)$ is a root of the denominator, we should have that

$$-\Psi(\beta)\,\mathbb{E}\,\underline{X}(T_\beta) = 1,$$

so that $\mathbb{E}\,\underline{X}(T_\beta) = -1/\Psi(\beta)$ (cf. also Lemma 1.1). From the above we conclude that, in agreement with the result stated in Theorem 1.1 for the transform of $\bar{X}(T_\beta)$,

$$\mathbb{E}\,e^{-\alpha Q(T_\beta)} = \frac{\alpha - \Psi(\beta)}{\varphi(\alpha)-\beta}\,\frac{\beta}{\Psi(\beta)}.$$

1.6 Method 4: Kolmogorov forward equations

As we did in the previous section, in this section we find the transform of $\bar{X}(T_\beta)$ by finding an expression for the transform of $Q(T_\beta)$ (and exploiting the duality property that was discussed in Section 1.2). Where we used the Kella-Whitt martingale in Section 1.5, we now rely on what could be called a 'small timestep argumentation', but which can be interpreted in terms of the classical Kolmogorov forward equations.

▷ *Transform of workload at an exponential epoch.* We start by deriving an expression involving the density and distribution of the queue workload, which we primarily include here for later reference; after that, we use a similar argumentation to characterize the transform of $Q(T_\beta)$.

Define $F_t(x)$ as the probability that $Q(t)$ does not exceed x, where $Q(0) = 0$, and let $f_t(x)$ denote the corresponding density. An elementary Δt-argument gives, up to $o(\Delta t)$-terms,

$$F_{t+\Delta t}(x) = F_t(x + r\Delta t)(1 - \lambda\Delta t) + \lambda\Delta t\left(\int_{0+}^{x} f_t(y)\,\mathbb{P}(B \leqslant x - y)\,\mathrm{d}y + F_t(0)\,\mathbb{P}(B \leqslant x)\right);$$

indeed, the first term on the right-hand side corresponds to the scenario of no arrival (so that the workload had to be at most $x + r\Delta t$ at time t), whereas the second term corresponds to the scenario with an arrival (with the corresponding job being sufficiently small to make sure the level remains below x at time $t + \Delta t$). Subtracting

$F_t(y + r\Delta t)$ from both sides, dividing by Δt and letting $\Delta t \downarrow 0$ leads to the integro-differential equation, for $y \geq 0$,

$$\frac{\partial}{\partial t}F_t(y) = r f_t(y) - \lambda F_t(y) + \lambda \int_{0+}^{y} f_t(x)\,\mathbb{P}(B \leq y - x)\,\mathrm{d}x + \lambda F_t(0)\,\mathbb{P}(B \leq y). \tag{1.8}$$

A similar reasoning can be followed to analyze the LST of $Q(t)$. To this end, we define the transforms

$$\kappa_t(\alpha) := \mathbb{E}\, e^{-\alpha Q(t)}, \quad \bar\kappa_t(\alpha) := \mathbb{E}\, e^{-\alpha Q(t)} 1\{Q(t) > 0\} = \kappa_t(\alpha) - \pi_t,$$

where $\pi_t := \mathbb{P}(Q(t) = 0) = F_t(0)$. Then, up to $o(\Delta t)$-terms,

$$\begin{aligned}\bar\kappa_{t+\Delta t}(\alpha) + \pi_{t+\Delta t} &= \kappa_{t+\Delta t}(\alpha)\\ &= \bar\kappa_t(\alpha)\left(1 - \lambda\Delta t + \lambda\Delta t\, b(\alpha) + r\alpha\Delta t\right) + \pi_t\left(1 - \lambda\Delta t + \lambda\Delta t\, b(\alpha)\right)\\ &= \bar\kappa_t(\alpha)\left(1 + \varphi(\alpha)\Delta t\right) + \left(1 - \lambda\Delta t + \lambda\Delta t\, b(\alpha)\right)\pi_t.\end{aligned}$$

We have thus obtained the following differential equation.

Lemma 1.4 *For any $\alpha, t > 0$,*

$$\frac{\partial}{\partial t}\bar\kappa_t(\alpha) + \frac{\mathrm{d}}{\mathrm{d}t}\pi_t = \varphi(\alpha)\,\bar\kappa_t(\alpha) - \pi_t\,\lambda\,(1 - b(\alpha)).$$

We proceed by considering the differential equation of Lemma 1.4 at an exponentially distributed time with mean $1/\beta > 0$ (rather than at the deterministic time t, that is), sampled independently of the compound Poisson process under consideration. Using the standard identity (cf. Appendix A.2)

$$\int_0^\infty \beta e^{-\beta t}\left(\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right)\mathrm{d}t = -\beta f(0) + \beta \int_0^\infty \beta e^{-\beta t} f(t)\,\mathrm{d}t, \tag{1.9}$$

we obtain that

$$\begin{aligned}-\beta\bar\kappa_0(\alpha) + \beta\int_0^\infty \beta e^{-\beta t}\bar\kappa_t(\alpha)\,\mathrm{d}t - \beta\pi_0 + \beta\int_0^\infty \beta e^{-\beta t}\pi_t\,\mathrm{d}t\\ = \varphi(\alpha)\int_0^\infty \beta e^{-\beta t}\bar\kappa_t(\alpha)\,\mathrm{d}t - \lambda(1 - b(\alpha))\int_0^\infty \beta e^{-\beta t}\pi_t\,\mathrm{d}t.\end{aligned}$$

Observe that, due to $Q(0) = 0$, we have that $\bar\kappa_0(\alpha) = 0$ and $\pi_0 = 1$. Rearranging the above equation this yields, in self-evident notation, using the definition of $\varphi(\alpha)$,

$$(\beta - \varphi(\alpha))\,\bar\kappa_{T_\beta}(\alpha) + (\beta - \varphi(\alpha) + r\alpha)\,\pi_{T_\beta} = \beta. \tag{1.10}$$

The next step is to observe that π_{T_β} can be identified by inserting $\alpha = \psi(\beta)$ in (1.10). We thus obtain

$$@)_V = \frac{V}{Ak^1V^o}•$$

Upon combining the above findings,

$$E4\ ^{U\&^1)_{V}^{o}} = \hat{\ })_V{}^1U^o \ ¸ \ @)_V = \frac{V}{V \quad i^1U^o} \quad \frac{V \quad i^1U^o \ ¸ \ AU}{V \quad i^1U^o} @)_V \ ¸ \ @)_V$$
$$= \frac{V \quad AU\,@)_V}{V \quad i^1U^o} = \frac{U \quad k^1V^o}{i^1U^o \quad V} \frac{V}{k^1V^o} -$$

thus (again using duality) recovering Theorem 1.1.

B *Transform of stationary workload, rate conservation.* Suppose that the net-profit condition $\_E \quad \ddot{Y}\,A$ is in place, implying stability of the dual queue. Then, as before, the transform of $\&^1 1^o$ (and hence also of $\bar{.}^1 1^o$) can be found from that of $\&^1)_V{}^o$ by letting $V\,\#\,0$. The above derivation, however, slightly simplifies in this stationary regime. In the first place, the integro-differential equation (1.8) becomes, for any $H \ i \ 0$,

$$0 = A\,5^1H^o \quad \_ \quad ^1H^o \ ¸ \ \_ \int_{0¸}^{1\ H} 5^1I^o\,P^1 \quad 6\ H \quad I^o\,dI \ ¸ \quad ^10^o\,P^1 \quad 6\ H^o \ -$$

with $5^1\,^o$ the density of the stationary workload (and hence also of $\bar{.}^1 1^o$) and $\ ^1\,^o$ the corresponding cumulative distribution function. This equation can alternatively be written as

$$A\,5^1H^o = \_ \int_{0¸}^{1\ H} 5^1I^o\,P^1 \quad i\ H \quad I^o\,dI \ ¸ \quad ^10^o\,P^1 \quad i\ H^o \ • \tag{1.11}$$

This identity has an appealing level-crossing interpretation (cf. Appendix B.1), known as ‘rate conservation’: the left-hand side can be considered as the probability flux through the level $H \ i \ 0$ from above, whereas the right-hand side represents the probability flux through the same level from below.

Define

$$\hat{\ }^1U^o := E4\ ^{U\&^1 1^o} - \quad \check{\ }^1U^o := E4\ ^{U\&^1 1^o} 1 f\&^1 1^o \ i \ 0g = \hat{\ }^1U^o \quad @-$$

where $@ := \quad ^10^o = P^1\&^1 1^o = 0^o$. By taking Laplace transforms in (1.11), we can prove that $\check{\ }^1U^o = @\_^1 1 \quad 1^1U^{oo} • i^1U^o$; see Exercise 1.4. Alternatively, by using the differential equation stated in Lemma 1.4, and equating the derivatives (with respect to time) to 0, we find the equation

$$0 = i^1U^o\,\check{\ }^1U^o \quad @\_\ 1 \quad 1^1U^o \ -$$

also leading to $\check{\ }^1U^o = @\_^1 1 \quad 1^1U^{oo} • i^1U^o$. Then it takes a few elementary calculations to verify that, indeed, $\hat{\ }^1U^o = U\,i^{0\,1}0^o • i^1U^o$, in line with what was stated in Corollary 1.1; in passing, we have also shown that

$$@ = \frac{i\,{}^0{}^1 0^{\text{o}}}{A} = 1 \quad \frac{\text{E}}{A}\text{-}$$

equalling the constant 2 that was defined in Section 1.3.

In Proposition 1.1 we stated that the all-time maximum $.^{\bar{}}{}^1 1^{\text{o}}$ is distributed as a geometric number of residual service times. This we can reprove using the fact that $.^{\bar{}}{}^1 1^{\text{o}}$ and $\&^1 1^{\text{o}}$ are equally distributed, in combination with the rate conservation equation that we stated above, as follows. With $6_8{}^1{}^{\text{o}}$ the density of the sum of 8 i.i.d. copies of the residual service time $^{\bar{}}$, and recalling that $^{\bar{}}$ has density $5\text{-}^1 C^{\text{o}} := \text{P}^1 \;\; > C^{\text{o}}\bullet\text{E}$, it is easily seen that we can rewrite the rate conservation equation (1.11) to

$$5\,^1 H^{\text{o}} = {}^1 1 \quad 2^{\text{o}} \int_{0_{\text{¸}}}^{1\;H} 5\,^1 I^{\text{o}}\, 6_1{}^1 H \quad I^{\text{o}}\, \mathrm{d}I \;\text{¸}\;\; {}^1 0^{\text{o}}\, 6_1{}^1 H^{\text{o}} \;\text{-}$$

which is a Volterra integral equation of the second kind. Iterating this relation once, it is immediately verified that

$$5\,^1 H^{\text{o}} = {}^1 1 \quad 2^{\text{o}} \;\; {}^1 0^{\text{o}}\, 6_1{}^1 H^{\text{o}} \;\text{¸}\; {}^1 1 \quad 2^{\text{o}2} \;\; {}^1 0^{\text{o}}\, 6_2{}^1 H^{\text{o}} \;\text{¸}\; {}^1 1 \quad 2^{\text{o}2} \int_{0_{\text{¸}}}^{1\;H} 5\,^1 I^{\text{o}}\, 6_2{}^1 H \quad I^{\text{o}}\, \mathrm{d}I\bullet$$

Continuing along these lines, and verifying that this so-called Picard iteration converges if 2 2 $^1 0\text{-}\, 1¼$, we find that

$$5\,^1 H^{\text{o}} = \sum_{8=1}^{\tilde{①}} {}^1 1 \quad 2^{\text{o}8}\, 2\, 6_8{}^1 H^{\text{o}}\text{-}$$

where we have used that $\;\; {}^1 0^{\text{o}} = 2$ due to the fact that the density $5\,^1{}^{\text{o}}$ should integrate to $1 \quad {}^1 0^{\text{o}}$. We conclude that, as desired, the stationary workload $\&^1 1^{\text{o}}$ is distributed as the sum of a geometric number (with success probability 2) of residual claim sizes, confirming the result stated in Proposition 1.1.

1.7 Discussion and bibliographical notes

In this concluding section we briefly discuss the relation between the material presented in this chapter and various branches of the literature. In addition, we provide a compact account of the relevant references.

B *Related branches of literature.* In the first place, the connection between the Cramér-Lundberg model and the corresponding M/G/1 queue has been observed (and exploited) before. We specifically refer to [4, Sections III.2 and VI.4] for a detailed account, or [22] for a more extensive treatment.

Secondly, in this monograph we consider the situation that the net cumulative claim process is a compound Poisson process minus a deterministic drift. Many of the results presented carry over to the more general setting of the net cumulative claim process being a *spectrally positive Lévy process*; Lévy processes are processes with stationary independent increments, and these are called spectrally positive in case they do not have any negative jumps. The compound Poisson process minus a linear drift is such a spectrally positive Lévy process. Another example of such a spectrally positive Lévy process is Brownian motion, which has no jumps (and hence no negative jumps). Yet another example is the Gamma process (minus a deterministic drift), which is a process with the remarkable feature that it exhibits almost surely infinitely many jumps in any finite time interval, therefore not classifying as a compound Poisson process (minus a deterministic drift). Textbooks on Lévy processes are e.g. [5, 24]. In [9, 18] there is a strong focus on the analysis of the extreme values attained by the Lévy process; see also [3, Ch. IX]. Treatments specifically covering Lévy applications in insurance risk are e.g. [4, 10, 19, 23].

In the third place, it is observed that c¹U- V°•V can be interpreted as the double Laplace transform of the time-dependent ruin probability ?¹D- C°• So as to numerically assess ?¹D- C°, a double numerical inversion is needed. Numerical Laplace inversion is a mature area of research, with [1, 16] being examples of relevant contributions in this context. In e.g. [2, 14] it is demonstrated that these computational methods have the potential to generate reliable output also when multidimensional inversion needs to be performed. In Section 3.4 we will point out how we can extend the analysis of this chapter so as to cover the ruin probability over a *phase-type* horizon; notably, phase-type distributions form a convenient class of distributions by which one can approximate any distribution on ¹0- 1° arbitrarily closely.

B *Literature on the conventional Cramér-Lundberg model.* We finish this chapter by providing selected pointers to relevant literature. Needless to say that one cannot be complete given the vast literature in this area. This motivates why we have restricted ourselves to

- mentioning (in our opinion) accessible introductions to the concepts discussed in this chapter;
- providing a few historic notes (acknowledging that our selection is by no means exhaustive);
- pointing out which elements of our exposition have *not* appeared earlier in the literature.

The Cramér-Lundberg model, introduced in Section 1.2, probabilistically describing the evolution of the surplus level, dates back about a century; see the seminal contributions [8, 20, 21]. As mentioned above, the duality between the Cramér-Lundberg model and the M/G/1 queue has been worked with before. We refer to Appendix B.1 and [4, Section X.5] for a compact account of the M/G/1 queue, and to [7] for an in-depth treatment.

Method 1 (Section 1.3), is, to the best of our knowledge, in principle novel. It evidently combines a variety of well-known ideas though, such as exponential killing, the memoryless property of the exponential distribution, some convenient properties of transforms, and conditioning on the first event.

Method 2 (Section 1.4) combines known and new elements. Lemma 1.2 could also have been derived using the classical fixed-point relation (sometimes referred to as the *Kendall functional equation*) that characterizes the busy-period transform, as was demonstrated in [9, Section VI.1]. Proposition 1.3 is essentially [22, Lemma 4, Section IV.10], but our proof is novel and does not require the Wiener-Hopf decomposition, while Proposition 1.4 is a slight modification of [22, Theorem 14, Section IV.10].

Method 3 (Section 1.5) relies on the Kella-Whitt martingale that was introduced in [17]; see also [18, Section IV.4] and [3, Section IX.3] for textbook treatments, including a formal proof of the process $'' {}^{1}C^{\circ}$ being a martingale.

Method 4 (Section 1.6) is based on the combination of various standard elements. It effectively relies on the Kolmogorov forward equations; see e.g. [25] for an early derivation along these lines. Level-crossing arguments have appeared broadly in the queueing and ruin literature. We refer to e.g. [4, Chapter VIII] and [6], also covering the case in which the stochastic dynamics depend on the current workload or reserve level; we get back to this class of models in great detail in Chapter 6. It is noted that, formally, in (1.8) one should prove the differentiability of $_{C}{}^{1}H^{\circ}$ with respect to C; we refer to [15] for a detailed discussion of such continuity and differentiability issues.

In practical contexts, one is typically not interested in just the ruin probability, but in various related measures as well. A relevant quantity in this context is the ruin time $g^{1}D^{\circ}$ (on the event that ruin occurs). In addition, for obvious reasons the insurance firm wants to have insight into the loss, or 'overshoot', $-_{D}{}^{1}g^{1}D^{\circ\circ}$ as well as the corresponding 'undershoot' $-_{D}{}^{1}g^{1}D^{\circ}\ {}^{\circ}$. As it turns out, relying on the techniques presented in the present chapter, the joint transform of these quantities can be derived with a modest amount of additional work. Pioneering papers in this area are by Gerber and Shiu [11, 12], but see also the compact exposition in [19]. In Exercise 1.2 it is pointed out how the above-mentioned joint transform can be evaluated. We get back to the evaluation of such detailed ruin-related metrics in Chapter 4, but then for a more general class of models than those covered by the present chapter.

1.8 Biographical sketches

B HARALD CRAMÉR (¢ Stockholm, Sweden, 1893 — y Stockholm, Sweden, 1985) was a Swedish mathematician, mainly known for his contributions to probability theory, mathematical statistics, and actuarial science. While being involved in chemical research in his early career, his PhD thesis was of a mathematical nature, with a focus on number theory. He then combined a job as an assistant professor at the University of Stockholm, in which he continued his research in number theory, with an appointment at the Svenska Life Assurance Company. Gradually his interest shifted into the direction of probability theory and its formal underpinnings. At the age of 36 he was appointed full professor of actuarial mathematics and mathematical statistics at the University of Stockholm.

Within probability theory Cramér is known for the introduction of various important, and original, concepts and results. He has been one of the pioneers of large deviations theory, a domain that focuses on quantifying rare-event probabilities. He proved an early version of what was later called *Cramér's theorem*, arguably the most fundamental large deviations result. It states that, under suitable conditions, the probability of the sample mean of a sequence of independent and identically distributed random variables exceeding some value (larger than the mean) decays exponentially. In risk theory, a subdomain within applied probability, he helped developing the *Cramér-Lundberg model*, which became a key concept within actuarial science. He succeeded to embed the model, which was originally proposed by Filip Lundberg, into the theory of stochastic processes. Cramér also gave his name to the *Cramér-Wold device*, a useful theorem that states that a sequence of random vectors converges in distribution to a limiting random vector if and only if this holds for any linear combination of the entries of the random vectors.

Cramér is generally regarded as one of the most influential probabilists and statisticians of the twentieth century, and one of the founding fathers of mathematical risk theory. Besides being a great scientist, he is also known as the PhD supervisor and mentor of a sequence of highly successful scholars, including Kai Lai Chung and Herman Wold. In addition, as a service to the community, he took on various management positions, including the presidency of the University of Stockholm.

B Filip Lundberg (¢ Uppsala, Sweden, 1876 — y Stockholm, Sweden, 1965) was a Swedish actuary and probabilist, known for his pioneering contributions to mathematical risk theory. Lundberg combined a business career with an academic career. In his appointments at various Swedish insurance firms he was inspired to set up a mathematical theory to evaluate and control collective risk. In a time in which the available probabilistic tools were still relatively poorly developed, he managed to set up the rudimentary versions of various actuarial concepts that are still used nowadays.

After having studied mathematics at the University of Uppsala, Lundberg wrote a PhD thesis at the interface of probability theory and collective risk, that was motivated by problems that he had encountered in his work in the insurance industry. He received his PhD degree in 1903, again from the University of Uppsala. Until the 1930s he contributed to actuarial science by a sequence of foundational papers.

Lundberg is probably best known for proposing the *Cramér-Lundberg model*. In this model, independent and identically distributed claims arrive according to a Poisson process, while the insurance firm's clients pay premiums at a constant rate. As a result, the firm's surplus level evolves as a compound Poisson model with positive drift (starting at some given initial capital). Although the Cramér-Lundberg model evidently oversimplifies various aspects of the surplus level dynamics, it has become a frequently used benchmark model. Later various generalizations were proposed that made the model more realistic. Lundberg in addition gave his name to the *Lundberg inequality*, a uniform upper bound on the probability of ultimate ruin, i.e., the probability of the reserve level ever dropping below zero. Only when Cramér and coworkers started to refer to Lundberg's contributions, and further developed them, his findings were brought to the attention of the international research community.

Besides having been an internationally renowned scientist, Lundberg played an important role in the insurance industry. Due to his strong opinions on policies to be used in the insurance business, he became an important spokesman and debater. In addition, he served as the chairman of the Association of Swedish Life Insurance Companies.

B FÉLIX POLLACZEK (¢ Vienna, Austria-Hungary, 1892 — y Boulogne-Billancourt, France, 1981) was an (Austrian-born) French engineer and mathematician, known for his many contributions to various fields of mathematics. Pollaczek studied mathematics at the Technical University of Vienna, received the MSc degree in electrical engineering from the Technical University of Brno in 1920, and then his PhD in mathematics from the University of Berlin in 1922. During World War I his education was interrupted to serve in the Austrian army.

In his early career Pollaczek, whose PhD thesis was supervised by Issai Schur, focused on problems deriving from number theory. During the next decades, however, he primarily worked in industry. After having spent two years as an engineer at AEG, he took on a position at the German Postal, Telephone, and Telegraph Services, both being located in Berlin. After the nazis seized power in Germany, all Jewish civil servants were forced out of government jobs, so that Pollaczek lost his appointment. He decided to flee to France, where he became a consultant for the Société d'Études pour Liaisons Téléphoniques et Télégraphiques. Later he moved to the Centre National de la Recherche Scientifique (CNRS).

Pollaczek's scientific work covered a broad range of mathematical subdisciplines, including number theory, mathematical analysis, mathematical physics, probability theory, and operations research. *Pollaczek polynomials*, a class of frequently used orthogonal polynomials, were named after him. He is perhaps most known, however, for his contributions to queueing theory. In particular, the celebrated *Pollaczek-Khinchine* formula, first published by Pollaczek in 1930, gives an explicit expression for the mean waiting time in the single-server queue fed by a Poisson stream of customers with an arbitrary service-time distribution. Later this result was extended so as to provide the full Laplace-Stieltjes transform of the waiting time. Through a duality relation between such queueing processes and the Cramér-Lundberg model, these results can also be used to evaluate the probability of ultimate ruin in the insurance context.

Importantly, during most of Pollaczek's career stochastic-process theory was still in its infancy, so that he could hardly rely on existing sophisticated machinery, which makes his achievements even more remarkable. In 1977 he was awarded the John von Neumann Theory Prize, for his fundamental and sustained contribution to theory in operations research and management science.

B ALEKSANDR YAKOVLEVICH KHINCHINE (¢ Kondrovo, Russian Empire, 1894 — y Moscow, Soviet Union, 1959) was a Russian mathematician, primarily known for his contributions to probability theory and information theory. Khinchine studied mathematics at the Moscow State University, where he became a full professor already at the age of 28.
In his early career he made several fundamental contributions to probability theory. In the first place, he established *Khinchine's weak law of large numbers*, providing conditions under which the sample mean of a sequence of random variables converges (in probability) to the mean. A second major result is the *law of the iterated logarithm*, which describes the behavior of sample means in the region between the law of large numbers and the central limit theorem. Thirdly, extending results by Norbert Wiener, Khinchine derived the *Wiener-Khinchine theorem*, which states that the autocorrelation function of a specific class of stationary random processes has a spectral decomposition that is given by the process' power spectrum. In his 1938 book *Limit distributions for the sum of independent random variables* Khinchine developed the theory of infinitely divisible and stable distributions, and their applications to the theory of limit distributions for sums of independent random variables. In a rare joint paper, Khinchine and Lévy gave the general formula for the characteristic function of a stable distribution.
In his exposition of Khinchine's work in mathematical probability (Annals of Mathematical Statistics **33** (1962), pp. 1227-1237), Harald Cramér discusses the tremendous development of probability as an established mathematical discipline between 1920 and 1940, expressing the expectation that 'future historians will ascribe this development, as far as the mathematical side of the subject is concerned, above all to the creative powers of three men: A.I. Khinchine, A.N. Kolmogorov, and P. Lévy.'
Queueing theory played an important role in Khinchine's work. Two years later than Pollaczek, Khinchine also found the formula for the mean waiting time in the single-server queue with Poisson arrivals and arbitrary service times. The distributional version of this relation, nowadays known as the *Pollaczek-Khinchine formula*, has found widespread use in actuarial science as well. The *Palm-Khinchine* theorem entails that a superposition of a large number of renewal streams converges, after an appropriate scaling, to a Poisson process, which is a result that is also frequently used in the queueing context.
In his late career, Khinchine increasingly focused on information theory, studying various concepts that were introduced by Claude Shannon. In 1939 Khinchine was elected to the Soviet Union's Academy of Sciences. He was in addition awarded the Stalin Prize and the Order of Lenin.

B ANDREY NIKOLAEVICH KOLMOGOROV (¢ Tambov, Russian Empire, 1903 — y Moscow, Soviet Union, 1987) was a Russian mathematician whose influence on modern mathematics can hardly be exaggerated. He made deep and lasting contributions to many areas of mathematics and had 82 PhD students according to the Mathematics Genealogy project; in probability theory these include Borovkov, Dobrushin, Dynkin, Gelfand, Gnedenko, Prohorov, Shiryaev and Sinai.

His parents were not married, and his father did not take part in his upbringing while his mother died while giving birth to him. His mother's sister brought him up. His exceptional mathematical talent became clear at a very early age, and he published his first paper at the age of 19. In 1925, still an undergraduate, he published eight papers. In 1931 Kolmogorov became a professor at Moscow University. While retaining that position, in 1938 he was appointed head of the new Department of Probability and Statistics of the Steklov Institute of the USSR Academy of Sciences.

Kolmogorov's contributions to topics like turbulence, logic, information theory, ergodic theory, cohomology, statistics (the Kolmogorov-Smirnov test) and dynamical systems (the KAM-theory) are extensively discussed in his obituary (Bulletin of the London Mathematical Society **22** (1990), pp. 31-100). Here we restrict ourselves to mentioning some of his achievements in probability theory. In Hilbert's famous list of challenging mathematical problems of 1900, Problem 6 was 'To treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part; in the first rank are the theory of probability and mechanics'. In 1933, Kolmogorov published his *Grundbegri e der Wahrscheinlichkeitsrechnung*, in which he built probability theory from axioms in a rigorous way. It had a huge impact on the development of probability theory, and turned it into a mature mathematical discipline, building on measure-theoretic foundations. Incidentally, Kolmogorov also completely solved Hilbert's 13th problem.

Kolmogorov made many important contributions to the topic of limit theorems, starting in 1925 with a joint paper with Khinchine that contains the three-series theorem and culminating in the Gnedenko-Kolmogorov book *Limit distributions for sums of independent random variables*. He extended Khinchine's law of the iterated logarithm from Bernoulli trials to general distributions, not necessarily identical. Kolmogorov was strongly interested in biological problems, and with Dmitriev he introduced the term branching process (in the same year as T.E. Harris). He also introduced the concept of fractional Brownian motion. Kolmogorov devoted much time to studying the smoothing and prediction of stochastic processes with stationary increments. This work had considerable military significance; in World War 2 he and Norbert Wiener independently developed similar approaches to the problem of predicting the position of moving objects. Finally we should mention Kolmogorov's contributions to Markov processes, and in particular his development of the Chapman-Kolmogorov equations that appear throughout this book.

B PAUL LÉVY (¢ Paris, France, 1886 — y Paris, France, 1971) was a French mathematician who has had a profound impact on the modern theory of stochastic processes. He showed his mathematical talent at an early age, publishing his first paper, on semi-convergent series, in 1905 as an undergraduate student of the École Polytechnique. He completed his PhD thesis in 1911 under supervision of Hadamard and Volterra, on the topic of integro-differential equations. In 1913 he was appointed Professor at the École Nationale des Mines, and from 1920 until 1959 he was Professor of Analysis at the École Polytechnique.

Mathematics was a natural choice for Lévy, as his grandfather and father also were mathematicians. His father wrote several textbooks, including two with Rouché (Rouché's theorem is used a few times in this book). One of the three children of Paul and his wife Suzanne also became a prominent mathematician; furthermore, she married the renowned mathematician Laurent Schwartz.

Lévy was advisor of five PhD students, including Michel Loève (who wrote a beautiful obituary of Lévy; Annals of Probability **1** (1973), pp. 1-18), Benoit Mandelbrot and Wolfgang Doeblin.

Throughout Lévy's long career, that was only interrupted by the two World Wars (he fought in the first and, being Jewish, he had to be in hiding for a while in the second), he wrote ten books and over 270 papers. The fact that he hardly attended conferences and that his publications were in French most likely delayed international recognition for his achievements, while in Bourbaki France he was initially hardly seen as a mathematician. However, in more recent years his amazing legacy has become clear.

In 1919 Lévy had been asked to give three lectures on the calculus of probabilities at the École Polytechnique — a request that would shape his future and that of probability theory. The realization that a mathematical probability theory was essentially nonexistent led to an amazing research effort on the topic of limit laws, which in the period 1919-1934 resulted in major contributions to the central limit theorem, stable laws (triggered by a question at one of the above-mentioned lectures), infinitely-divisible distributions, the zero-one law, and distance metrics in the space of probability distributions. He also turned characteristic functions into a powerful tool for proving limit laws, via his continuity theorem.

The period 1934-1940 was even more fruitful. In his *annus mirabilis* 1934 he developed the theory of what is now called Lévy processes, and in that same year he also introduced the concept of what has become known as martingales. He also made important contributions to Brownian motion, one of his major accomplishments being the introduction of the concept of local time.

Lévy was an individualist, working alone and hardly reading work of others. As a consequence, he sometimes rediscovered known results. However, it also led him to profoundly original work. Lévy himself once said, with characteristic modesty: 'I am asking myself a not too difficult problem, so as not to break my teeth in front of an excess of difficulties, but all the same to have to make a big effort which occupies me and gives me the satisfaction of finding something.' Well, he definitely did find something!

Exercises

1.1 In this exercise we devise a recursive procedure to compute the moments of $\bar{Y}(T_\beta)$, i.e., the running maximum of $Y(t)$ over an exponentially distributed horizon. To this end, define

$$\mu_k := \mathbb{E}[(\bar{Y}(T_\beta))^k], \quad \varphi_k(\alpha) := \frac{\mathrm{d}^k}{\mathrm{d}\alpha^k}\varphi(\alpha) - 1\{k=0\}\beta.$$

(i) Prove, using Theorem 1.1, that

$$\sum_{k=0}^{\infty}\frac{\partial^k}{\partial\alpha^k} r(\alpha,\beta)\,\varphi_k(\alpha) = 1\{k=1\} + (\alpha - \psi(\beta))1\{k=0\}\,\frac{\beta}{\psi(\beta)}.$$

(ii) Show that (i) implies that

$$\sum_{k=0}^{\infty}(-1)^k\mu_k\,\varphi_k(0) = 1\{k=1\} - \psi(\beta)1\{k=0\}\,\frac{\beta}{\psi(\beta)}.$$

(iii) Show that this leads to the following recursion: $\mu_0 = 1$, while for $k \in \mathbb{N}$,

$$\mu_k = \frac{1}{\beta}\sum_{\ell=0}^{k-1}\binom{k}{\ell}(-1)^{k-\ell}\mu_\ell\,\varphi_{k-\ell}(0) - 1\{k=1\}\frac{(-1)^k}{\psi(\beta)}.$$

(iv) Compute μ_1 and μ_2.

1.2 Recall that we define the *ruin time* by

$$\tau(u) := \inf\{t \geq 0 : X_u(t) < 0\},$$

so that $p(u,t) = \mathbb{P}(\tau(u) \leq t)$. In this exercise we focus on evaluating the (transform of the) ruin probability, jointly with the time of ruin $\tau(u)$, the value of the reserve process *immediately before* ruin $X_u(\tau(u)-)$, and the value of the reserve process *at* ruin $X_u(\tau(u))$. The quantity $X_u(\tau(u)-) > 0$ is often referred to as the *undershoot*, whereas $-X_u(\tau(u)) \geq 0$ represents the corresponding *overshoot*. To analyze the joint transform, we define, with $\gamma := (\gamma_1,\gamma_2,\gamma_3)$,

$$p(u,t,\gamma) := \mathbb{E}\, e^{-\gamma_1\tau(u) - \gamma_2 X_u(\tau(u)-) - \gamma_3 X_u(\tau(u))}1\{\tau(u) \leq t\},$$

and let T_β be an exponentially distributed time with parameter β sampled independently from everything else; as $\tau(u)$ and $X_u(\tau(u)-)$ are non-negative whereas $X_u(\tau(u))$ is non-positive, we let $\gamma_1,\gamma_2 > 0$ and $\gamma_3 \leq 0$. Our objective is to step-by-step evaluate the transform with respect to u and t:

$$\pi(\alpha,\beta,\gamma) := \int_0^\infty\int_0^\infty e^{-\alpha u}\beta e^{-\beta t}p(u,t,\gamma)\,\mathrm{d}t\,\mathrm{d}u = \int_0^\infty e^{-\alpha u}p(u,T_\beta,\gamma)\,\mathrm{d}u.$$

For conciseness, below we write

?¹Dº ?¹D-)V-γº = E 4 W1g¹Dº W2 -D¹g¹Dº º W3 -D¹g¹Dºº 1fg¹Dº 6)Vg •

(i) Prove that, as C # 0, up to a >¹ Cº-term,

?¹Dº = 4 W1 C _ C ∫₀^D P¹ 2 dEº ?¹D Eº ¸

_ C ∫_D^1 P¹ 2 dEº 4 W2D 4 W3¹D Eº ¸ ¹1 _ C V Cº ?¹D ¸ A Cº •

(ii) The next step is to convert the equation found under (i) into an integro-differential equation. Show that, with I := W1 ¸ _ ¸ V,

A ?⁰¹Dº = _ ∫₀^D P¹ 2 dEº ?¹D Eº ¸

_ ∫_D^1 P¹ 2 dEº 4 W2D 4 W3¹D Eº I ?¹Dº•

(iii) We now multiply the entire equation by U4 UD and integrate over D. To this end, first show that, with c̄¹Uº c̄¹U- V- γº := Uc¹U- V- γº,

∫₀^1 ?⁰¹Dº U4 UD dD = U ?¹0º c̄¹Uº •

(iv) Prove the relation

AU ?¹0º c̄¹Uº = _1¹Uºc̄¹Uº ¸ _U (1¹ W3º 1¹U ¸ W2º) / (U ¸ W2 ¸ W3) I c̄¹Uº•

(v) Solve this equation to obtain, with c¹Uº c¹U- V- γº,

c¹Uº = 1 / (i ¹Uº W1 V) · A ?¹0º _ (1¹ W3º 1¹U ¸ W2º) / (U ¸ W2 ¸ W3) •

(vi) The only unknown constant left is ?¹0º• First find this constant, and then conclude that

c¹Uº = _ / (i ¹Uº W1 V) · (1¹ W3º 1¹k¹W1 ¸ Vº ¸ W2º) / (k¹W1 ¸ Vº ¸ W2 ¸ W3) (1¹ W3º 1¹U ¸ W2º) / (U ¸ W2 ¸ W3) •

(vii) Explain why in the expression for c¹U- V- γº the parameters V and W1 only appear as their sum (i.e., as W1 ¸ V).

1.3 Consider the workload process &¹Cº (with &¹0º = 0). Assume that the net-profit condition _ E Ÿ A is in place. Recall the regenerative formula, for a function 5¹ º and a stochastic process /¹Cº,

$$\mathbb{E}f(Z(\infty)) = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t f(Z(s))\,\mathrm{d}s = \frac{1}{\mathbb{E}T}\mathbb{E}\int_0^T f(Z(s))\,\mathrm{d}s,$$

with T a time between two successive regenerations of the process $Z(t)$; see e.g. [3, Ch. VI]. Let the stopping time T correspond to the end of the first busy period, where $Q(t)$ regenerates. As a consequence,

$$\frac{1}{\mathbb{E}T}\mathbb{E}\int_0^T e^{-\alpha Q(s)}\,\mathrm{d}s = \mathbb{E}e^{-\alpha Q(\infty)}.$$

(i) Prove that $\mathbb{E}Y(T) = -r/\lambda$.
(ii) Show that

$$\mathbb{E}T = \frac{1/\lambda}{1-\lambda\mathbb{E}B/r}.$$

(*Hint*: argue that $\mathbb{E}T = 1/\lambda + (\lambda\mathbb{E}B\,\mathbb{E}T)/r$, with $\lambda\mathbb{E}B\,\mathbb{E}T$ to be interpreted as the amount of work arriving in a busy period.)
(iii) Use the Kella-Whitt martingale of Section 1.5 to prove that

$$\mathbb{E}e^{-\alpha Q(\infty)} = \frac{\alpha\varphi'(0)}{\varphi(\alpha)}.$$

1.4 Take Laplace transforms in Equation (1.11) to show that

$$\kappa(\alpha) = q\lambda\frac{1-b(\alpha)}{\varphi(\alpha)}.$$

1.5 $(\ast)$ In Section 1.7 it was mentioned that the results of the present chapter carry over to $Y(t)$ being a spectrally positive Lévy process. In this exercise we consider a special case of such processes, namely the one in which $Y(t)$ is a compound Poisson process with drift (characterized by the usual λ, $b(\alpha)$, and r) perturbed by a Brownian motion (or: Wiener process). Concretely, the net cumulative claim process is

$$Y(t) := Y_W(t) + \sum_{i=1}^{N(t)} X_i,$$

where $Y_W(t) = -rt + \sigma W(t)$, with $W(t)$ a standard Brownian motion, and r and σ being two positive parameters. The focus is on all-time ruin, i.e., we wish to evaluate the transform of $\bar{Y}(\infty)$. We assume the net-profit condition $\lambda\mathbb{E}B < r$ holds.
(i) Prove that, for $\alpha > 0$,

$$\varphi_W(\alpha) := \log\mathbb{E}e^{-\alpha Y_W(1)} = \frac{\sigma^2\alpha^2}{2} + r\alpha.$$

(ii) Prove, using the same technique as used in the proof of Lemma 1.1, that $\bar{Y}_W(T_\beta)$ has an exponential distribution with parameter

$$G_{\check{V}} := \frac{A \,¸\, F}{f^2}-$$

and that $\underline{.}_{\,,}\ ^1)_V{}^{o}$ has an exponential distribution with parameter

$$G_V := \frac{A \,¸\, F}{f^2}-$$

where $F := \sqrt{A^2 \,¸\, 2Vf^2}$. Verify that $G_{\check{V}}\, G_V = 2V• f^2$.
(iii) Recall from Section 1.3 that a time-reversibility argument yields that $\bar{.}_{\,,}\ ^1)_V{}^{o}$ $._{\,,}\ ^1)_V{}^{o}$ and $\underline{.}_{\,,}\ ^1)_V{}^{o}$ are identically distributed (where it is left to the reader to verify that this argument still applies in the case of Brownian motion). Compute the transform of $._{\,,}\ ^1)_V{}^{o}$ in terms of $i_{\,,}\ ^1U^{o}$, and use this transform to prove that $\bar{.}_{\,,}\ ^1)_V{}^{o}$ and $\bar{.}_{\,,}\ ^1)_V{}^{o}$ $._{\,,}\ ^1)_V{}^{o}$ are independent.
(iv) In order to compute $\mathrm{E}4^{\ U.\bar{}\,^1 1^{o}}$, we define an alternative compound Poisson process $.\ ^1C^{o}$. It is characterized by the claim arrival rate $\_ := G\_$, claim sizes having the LST

$$1\ ^1U^{o} := \frac{G›\_}{G›\_ \,¸\, U}\, 1^1U^{o}-$$

and the premium rate being given by $A := 1$; let the corresponding Laplace exponent be denoted by $i\ ^1U^{o} := A\ U\ \ \_\ ^1 1\ \ 1\ ^1U^{oo}•$ Argue that

$$\mathrm{E}4^{\ U.\bar{}\,^1 1^{o}} = \frac{G›\_}{G›\_ \,¸\, U}\ r\ ^1U^{o}-$$

where

$$r\ ^1U^{o} = \frac{U\,^1 i\ ^{o0}1 0^{o}}{i\ ^1U^{o}}•$$

(*Hint*: see Figure 1.5.)
(v) Show that, with $i\,^1U^{o} := i_{\,,}\ ^1U^{o}\ \ \_^1 1\ \ 1^1U^{oo}$,

$$\mathrm{E}4^{\ U.\bar{}\,^1 1^{o}} = \frac{U\ i^{\,0 1}0^{o}}{i\,^1U^{o}}-$$

i.e., Corollary 1.1 carries over to the case that Brownian motion has been added to the Cramér-Lundberg model.

1.6 $^1¢^{o}$ In this exercise we analyze $?^1D^{o}\ \ ?^1D-)_V-\gamma^{o}$, as defined in Exercise 1.2, for the model defined in Exercise 1.5, i.e., the model in which the net cumulative claim process $.\ ^1C^{o}$ contains, besides a compound Poisson component, also a Brownian component $._{\,,}\ ^1C^{o}$. Observe that the level D can be first exceeded either at a claim arrival, thus leading to positive undershoot and overshoot, or between claim arrivals (i.e., due to the Brownian component), thus leading to the undershoot and overshoot both being equal to zero; as a consequence, the approach of Exercise 1.2 cannot be (directly) applied here. We refer to Figure 1.6 for a pictorial illustration of both cases. In this exercise we do not impose the requirement $A\ \ i\ \ 0•$

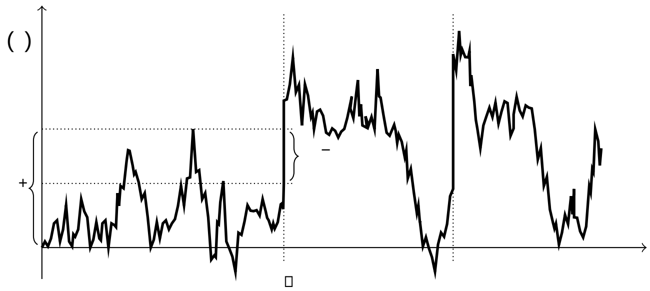

Fig. 1.5 Net cumulative claim process . ¹Cº for the sum of a compound Poisson process and a Brownian motion, as studied in Exercise 1.5, with the vertical dotted lines corresponding to claim arrivals. Here / › is distributed as .¯ , ¹)_º, which is exponential with parameter G›. Also, / is distributed as ._ , ¹)_º, which is exponential with parameter G_ , and independent of / ›.

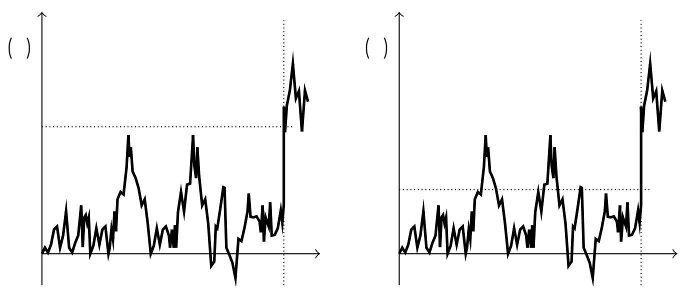

Fig. 1.6 Net cumulative claim process . ¹Cº for the sum of a compound Poisson process and a Brownian motion, as studied in Exercise 1.6, with a claim arrival at the vertical dotted line. Left panel: level D is exceeded due to a jump of the compound Poisson component. Right panel: level D is exceeded due to the Brownian component, i.e., with zero undershoot and overshoot.

Define ? ¹Dº as the counterpart of ?¹Dº, but with a claim arriving at time 0. In addition,

g , ¹Dº := inf fC ¡ 0 : . , ¹Cº ¡ Dg•

(i) Argue, by conditioning on the maximum of the Brownian component until the first claim arrival, that

?¹Dº = ¹0 ¹ ¹ ¹ ¹ ¹ P¹g , ¹Dº 2 dEº P¹)_¸V > Eº 4 W₁E ¸

C=0 F=0 E=0 D _4 ¹_¸VºC P¹.¯ , ¹Cº 2 dEº

P¹ ._ , ¹Cº 2 dFº 4 W₁C ? ¹D E ¸ Fº dC-

and

$$? \ (D) = \int_0^D P(\ \in dE)\, ?(D \ E) + \int_D^\infty P(\ \in dE)\, 4^{W_2 D}\, 4^{W_3(D\ E)}. \qquad (1.12)$$

(ii) Define $c\ (U)$ analogously to $c(U)$, but with $?(D)$ replaced by $?\ (D)$. Use (1.12) to verify that

$$c\ (U) = 1(U)\, c(U) + \ (U), \qquad (U) := \frac{1(\ W_3) \ \ 1(U + W_2)}{U + W_2 + W_3}.$$

(iii) Show that, with $\backslash := \_ + V + W_1$,

$$\int_0^\infty P(g_{,}(D) \in dE)\, P()_{\_+V} > E)\, 4^{W_1 E} = P(\ \bar{.}_{,}(\ )\backslash) > D),$$

so that, with $G_\backslash$ as defined in Exercise 1.5,

$$\int_0^\infty 4^{UD} \int_0^\infty P(g_{,}(D) \in dE)\, P()_{\_+V} > E)\, 4^{W_1 E}\, dD = \frac{1}{U + G_\backslash}.$$

(iv) Prove that

$$\int_{C=0}^\infty \int_{F=0}^\infty \int_{E=0}^D \_ 4^{(\_+V)C}\, P(\bar{.}_{,}(C) \in dE)\, P(\ \underline{.}_{,}(C) \in dF)\, 4^{W_1 C}\, ?(D \ E + F)\, dC$$
$$= \frac{\_}{\backslash} \int_{F=0}^\infty \int_{E=0}^D P(\bar{.}_{,}()\backslash) \in dE)\, P(\ \underline{.}_{,}()\backslash) \in dF)\, ?(D \ E + F),$$

and that in addition

$$\frac{\_}{\backslash} \int_{D=0}^\infty 4^{UD} \int_{F=0}^\infty \int_{E=0}^D P(\bar{.}_{,}()\backslash) \in dE)\, P(\ \underline{.}_{,}()\backslash) \in dF)\, ?(D \ E + F)\, dD$$
$$= \frac{\_}{\backslash} G_\backslash\, E\, 4^{U\bar{.}_{,}()\backslash)} \frac{c(G_\backslash) \ \ c(U)}{U \ \ G_\backslash}$$
$$= \frac{\_}{i_{,}(U) \ \ \backslash} \left(c(G_\backslash) \ \ c(U)\right).$$

(The latter computation is a tedious one!).

(v) Upon combining the above, show that

$$c(U) = \frac{(U \ \ G_\backslash)(f^2/2 + \_(c(G_\backslash) \ \ 1(U)\, c(U) \ \ \ (U)))}{i_{,}(U) \ \ \backslash},$$

and express $c(U)$ in terms of the model primitives and $c(G_\backslash)$.

(vi) The last step is to identify the constant $c(G_\backslash)$. Show that, with $k(\ )$ denoting the right inverse of $i(\ )$, $c(G_\backslash)$ satisfies

$$k(W_1 + V)\ \ G_\backslash \ \ f^2/2 + \_\ c(G_\backslash) \ \ \ (W_1 + V) = 0.$$

(vii) Conclude that

$$c(U) = \frac{(U \quad k(W_1 + V)) f^2 \bullet 2 + \_ \; (\quad (k(W_1 + V)) \quad (U))}{i(U) \quad W_1 \quad V}-$$

and verify that

$$\int_0^\infty 4^{\;UD}\, \mathrm{P}(\bar{.}(1)_V) \; i \; D)\, \mathrm{d}D = \frac{1}{i(U) \quad V} \quad U \quad k(V) \; \frac{f^2}{2} + \frac{i(U)}{U} \quad \frac{V}{k(V)} \; ;$$

cf. (1.4).
(viii) The remainder of this exercise considers the overshoot $-(g(D))$. We focus on the case that $i^{0}(0) = A \quad \_ \mathrm{E} \quad \ddot{Y}\, 0$ so that ruin eventually happens. Verify that, for $U > 0$ and $W_3 \; 6 \; 0$,

$$\int_0^\infty U4^{\;UD}\, \mathrm{E}\, 4^{\;W_3 \, -_D (g(D))}\, \mathrm{d}D = \frac{U}{i(U)} \quad U \quad k(0) \; \frac{f^2}{2} + \_ \quad (k(0)) \quad (U) \; \bullet$$

(ix) Observe that $-_D(g(D))$ has an atom in zero, due to the fact that ruin can possibly be reached 'due to the Brownian component' (i.e., between two subsequent claims). Show that

$$\int_0^\infty U4^{\;UD}\, \mathrm{P}(-_D(g(D)) = 0)\, \mathrm{d}D = \frac{U}{i(U)} \; (U \quad k(0)) \frac{f^2}{2} \; \bullet$$

(x) Verify that

$$\lim_{D \,!\, 1} \mathrm{P}(-_D(g(D)) = 0) = \frac{k(0) \; f^2}{2\, i^{0}(0)} \bullet \tag{1.13}$$

In addition, argue that the expression in the right-hand side of (1.13) is indeed between 0 and 1. (*Hint*: consider $i(U)$ for $U = k(0)$ and use $1(U) > 1 \quad U\mathrm{E} \;\;$.)

1.7 When considering the workload process, so far we have always assumed that the system starts empty, i.e., $\&(0) = 0$. In this exercise we show how to extend our results to the case that $\&(0) = G$ for arbitrary $G > 0\bullet$
(i) Prove that, for any $C > 0$,

$$\&(C) = .\,(C) + \max\mathrm{f}G- \; \_(C)\mathrm{g}\bullet$$

(ii) Use the Wiener-Hopf factorization, and the expression presented in part (i), to establish that

$$\mathrm{E} \; 4^{\;U\&(1)_V} \,\mathrm{j}\, \&(0) = G \; = \mathrm{E}\, 4^{\;U.\bar{}(1)_V} \int_0^\infty k(V) 4^{\;k(V)E}\, 4^{UE}\, 4^{\;U\max\mathrm{f}G\text{-}E\mathrm{g}}\, \mathrm{d}E\bullet$$

(iii) Show that

$$\mathrm{E} \; 4^{\;U\&(1)_V} \,\mathrm{j}\, \&(0) = G \; = \frac{V}{i(U) \quad V} \; \frac{U}{k(V)} 4^{\;k(V)G} \quad 4^{\;UG} \; \bullet \tag{1.14}$$

1.8 In this exercise we develop an alternative proof for (1.14).
(i) Argue that, with $\sigma(x)$ as defined in the proof of Lemma 1.1,

$$\mathbb{E}\, e^{-\alpha Y(T_\beta)} \,|\, Y(0) = x = e^{-\alpha x}\, \mathbb{E}\, e^{-\alpha X(T_\beta)} 1\{T_\beta \geq \sigma(x)\} +$$
$$\mathbb{P}(\sigma(x) < T_\beta)\, \mathbb{E}\, e^{-\alpha Y(T_\beta)} \,|\, Y(0) = 0.$$

(ii) Use

$$\mathbb{E} e^{-\alpha X(T_\beta)} = \mathbb{E}\, e^{-\alpha X(T_\beta)} 1\{T_\beta \geq \sigma(x)\} + \mathbb{E}\, e^{-\alpha X(T_\beta)} 1\{T_\beta < \sigma(x)\}$$

to prove that

$$\mathbb{E}\, e^{-\alpha X(T_\beta)} 1\{T_\beta \geq \sigma(x)\} = \left(1 - e^{(\alpha - \psi(\beta))x}\right) \frac{\beta}{\beta - \varphi(\alpha)}.$$

(iii) Use the above findings to derive (1.14).

1.9 Consider the case that B is exponentially distributed with parameter μ.
(i) Invert the $\kappa(\alpha, \beta)$ expression in (1.4) with respect to α to show that $p(u, T_\beta)$ has, for coefficients $\gamma_1(\beta), \ldots, \gamma_4(\beta)$ and exponents $\alpha_1(\beta)$ and $\alpha_2(\beta)$, the following form:

$$p(u, T_\beta) = \gamma_1(\beta) e^{\alpha_1(\beta) u} + \gamma_2(\beta) e^{\alpha_2(\beta) u} - \frac{\beta}{\psi(\beta)} \left(\gamma_3(\beta) e^{\alpha_1(\beta) u} + \gamma_4(\beta) e^{\alpha_2(\beta) u}\right).$$

Determine $\alpha_1(\beta)$, $\alpha_2(\beta)$ and $\gamma_i(\beta)$, $i = 1, \ldots, 4$.
(ii) Assume from now on that $\lambda/(r\mu) < 1$, so that $\bar{X}(\infty)$ is finite almost surely. Recall the definition of the ruin time $\tau(u)$ from Remark 1.3. Use the above expression for $p(u, T_\beta)$ to obtain $\mathbb{E}(\tau(u) 1\{\tau(u) < \infty\})$.
(iii) Letting $\beta \downarrow 0$, show that

$$p(u) = \frac{\lambda}{r\mu} e^{-(\mu - \lambda/r)u}.$$

(*Hint:* recall that $1/\psi'(0) = \varphi'(0)$.)
(iv) Prove the result of (iii) also by determining $\pi(\alpha)$ and inverting it.

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Chapter 2
Asymptotics

Abstract — This chapter provides asymptotic expressions for the ruin probability in the regime that the initial reserve level u grows large. One needs to distinguish between the case that the claim sizes are light-tailed, in which the ruin probability decays essentially exponentially, and the case that the claim sizes are heavy-tailed, in which the ruin probability decays as the complementary distribution function of a residual claim size. While the focus in this chapter is on the asymptotics of the all-time ruin probability $\psi(u)$, we also briefly discuss the asymptotics of its finite-time counterpart $\psi(u,t)$. We conclude this chapter by some comments on another limiting regime, corresponding to the heavy-traffic scaling in queueing theory.

2.1 Introduction

A disadvantage of the results presented in Chapter 1 is that they are in terms of transforms. As mentioned, these can be numerically inverted, but one lacks generic insight into the behavior of the ruin probability as a function of the initial reserve level u. However, if one settles for just its *tail* (which makes sense, given that ruin should occur rarely), rather explicit asymptotic results can be derived. This chapter analyzes the ruin probability in the regime that u grows large.

We throughout assume that $\mathbb{E}\,A(1) < 0$, to make sure that the net cumulative claim process $A(t)$ does not drift to ∞ as $t \to \infty$, and that consequently $\psi(u) \to 0$ as $u \to \infty$. This is equivalent to assuming that the net-profit condition $\lambda\,\mathbb{E}\,B < r$, as was introduced in Remark 1.1, holds.

When considering tail asymptotics, it turns out that one needs to distinguish between the claim-size distribution having a light or heavy tail, displaying intrinsically different behavior; cf. Figure 2.1.

In the light-tailed case, $\psi(u)$ decays exponentially, with (for large u) the net cumulative claim process moving 'roughly gradually' towards level u. Put differently, when 'zooming out', the path to a high level u increasingly looks like a straight line.

In the heavy-tailed ('subexponential') case, however, exceeding level u is (for large u) with overwhelming probability due to a single large claim, and as a consequence $p(u)$ behaves as the tail of the residual claim size distribution function.

The light-tailed and heavy-tailed cases are dealt with in Sections 2.2 and 2.3, respectively.

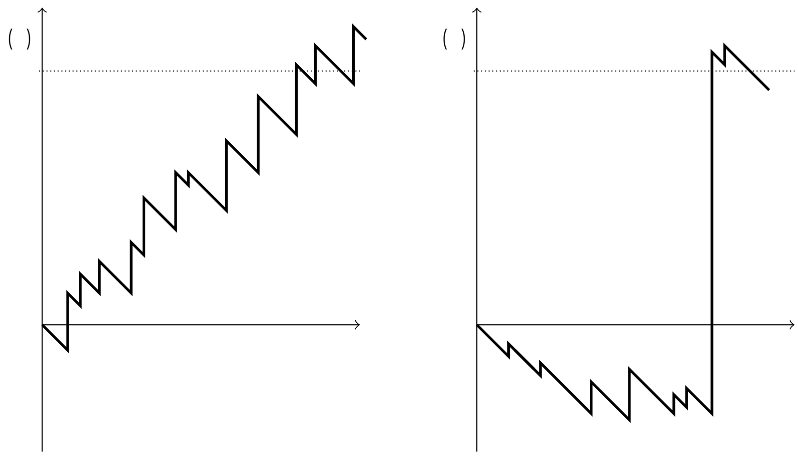

Fig. 2.1 A typical trajectory of the net cumulative claim process $Y(t)$ exceeding a high level u in the light-tailed case (left panel), and in the heavy-tailed case (right panel).

We conclude this chapter by briefly touching upon a few related topics. In the first place, in Section 2.4 we comment on the asymptotics of the time-dependent ruin probability $p(u,t)$. In addition, in Section 2.5 we discuss asymptotics in another limiting regime, in the queueing literature usually referred to as the heavy-traffic regime, in which $\epsilon = 1 - \lambda\,\mathbb{E}B/r$ is sent to 0 (from above, that is).

2.2 Light-tailed case

In this section we assume that there is a strictly positive solution $\alpha^\star$ to the equation $\varphi(-\alpha^\star) = 0$, where we recall that $\varphi(\alpha) := \log \mathbb{E}e^{-\alpha Y(1)} = r\alpha - \lambda(1 - b(\alpha))$. As the existence of this solution means that $\mathbb{E}e^{-\alpha Y(1)}$ is finite for some negative value of α, we observe that in this setting all moments of $Y(1)$ should be finite (and hence also all moments of $Y(t)$ for any $t > 0$). This explains why we refer to this setting as the *light-tailed case*. It implicitly means that the claim size B is light-tailed as well; we write $B \in \mathscr{L}$. The primary objective of this section is to identify the *exact asymptotics* of $p(u)$ in the situation that $B \in \mathscr{L}$: we find an explicit function $f(u)$ such that $p(u)/f(u) \to 1$ as $u \to \infty$.

▷ *Change-of-measure.* In our analysis an important role is played by the concept of *change-of-measure*. In this construction, we work with a compound Poisson process with drift that does not have Laplace exponent $\varphi(\alpha)$ but rather $\varphi_{\mathbb{Q}}(\alpha) := \varphi(\alpha - \theta^\star)$; we consistently use the symbol $\mathbb{Q}$ to denote the probability measure that goes with this alternative Laplace exponent. To see that $\varphi_{\mathbb{Q}}(\alpha)$ is indeed a Laplace exponent of a compound Poisson process with drift, first observe that, due to the fact that $\theta^\star$ solves the equation $r\theta^\star - \lambda(1 - b(-\theta^\star)) = 0$, we can write

$$\begin{aligned}\varphi(\alpha - \theta^\star) &= r(\alpha - \theta^\star) - \lambda(1 - b(\alpha - \theta^\star)) \\ &= r\alpha - \lambda b(-\theta^\star)\left(1 - \frac{b(\alpha - \theta^\star)}{b(-\theta^\star)}\right).\end{aligned}$$

Now note that $\mathbb{P}(B \in \mathrm{d}x)\, e^{\theta^\star x}/b(-\theta^\star)$ is a density (i.e., non-negative and integrating to 1), corresponding to a random variable with the Laplace-Stieltjes transform $b(\alpha - \theta^\star)/b(-\theta^\star)$; we say that this density is an *exponentially twisted* version of the original density. Combining the above, we conclude that $\varphi_{\mathbb{Q}}(\alpha) = \varphi(\alpha - \theta^\star)$ is the Laplace exponent of a compound Poisson process where the claim arrival rate is $\lambda_{\mathbb{Q}} := \lambda b(-\theta^\star)$, the claims have Laplace-Stieltjes transform $b_{\mathbb{Q}}(\alpha) := b(\alpha - \theta^\star)/b(-\theta^\star)$, while the negative drift remains r.

Recall that the process $Y(t)$ drifts to $-\infty$ under the original measure $\mathbb{P}$, due to the assumption $\mathbb{E}Y(1) < 0$. We now investigate whether this property has changed under $\mathbb{Q}$. To this end, first note that the mean of the claim sizes under $\mathbb{Q}$ is

$$\mathbb{E}_{\mathbb{Q}} B = -b_{\mathbb{Q}}'(0) = -\frac{b'(-\theta^\star)}{b(-\theta^\star)},$$

with $\mathbb{E}_{\mathbb{Q}}(\cdot)$ denoting expectation under $\mathbb{Q}$. As a consequence,

$$\mathbb{E}_{\mathbb{Q}} Y(1) = \lambda_{\mathbb{Q}}\left(-\frac{b'(-\theta^\star)}{b(-\theta^\star)}\right) - r = -\lambda b'(-\theta^\star) - r = -\varphi'(-\theta^\star) = -\varphi_{\mathbb{Q}}'(0), \tag{2.1}$$

which is positive due to the convexity of $\varphi(\cdot)$ and the definition of $\theta^\star$. Conclude that under $\mathbb{Q}$ the process $Y(\cdot)$ drifts to ∞. We refer to Figure 2.2 for a pictorial illustration.

▷ *Expression for ruin probability using* $\mathbb{Q}$. In this section, we denote by $\tau(u)$ the first time that $Y(\cdot)$ enters the set (u, ∞). As a consequence, the probability of our interest, $p(u)$ can be written as $\mathbb{P}(\tau(u) < \infty)$; obviously, due to $\mathbb{E}Y_1 < 0$, this probability vanishes as $u \to \infty$. The main idea behind our derivation is to perform the experiment of verifying whether or not $\tau(u) < \infty$ applies under $\mathbb{Q}$ rather than under $\mathbb{P}$. Under $\mathbb{Q}$ the event $\{\tau(u) < \infty\}$ has probability 1, due to $\mathbb{E}_{\mathbb{Q}}Y(1) > 0$, but a 'compensation' is applied to correct for the difference between the two probability measures.

More concretely, let N be the index of the claim at which, in our experiment, the set $[u, \infty)$ has been reached. This means that at that point the interarrival times (say) $\boldsymbol{A} \equiv (A_1, \ldots, A_N)$ and claim sizes $\boldsymbol{B} \equiv (B_1, \ldots, B_N)$ have been sampled (under $\mathbb{Q}$, that is). With $L \equiv L(\boldsymbol{A}, \boldsymbol{B})$ denoting the likelihood ratio of the sampled $(\boldsymbol{A}, \boldsymbol{B})$

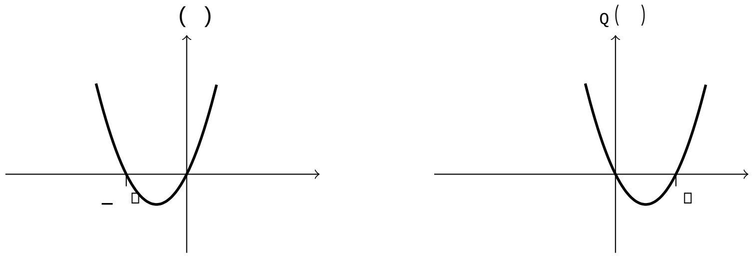

Fig. 2.2 The functions $\psi(\theta)$ (left panel) and $\psi_Q(\theta)$ (right panel). Observe that $\psi'(0) > 0$ but $\psi_Q'(0) < 0$.

(under P, relative to Q), we have the identity

$$p(u) = \mathbb{P}(\tau(u) < \infty) = \mathbb{E}\,1\{\tau(u) < \infty\} = \mathbb{E}_Q\, 1\{\tau(u) < \infty\}\, L(\boldsymbol{E}, \boldsymbol{B}). \qquad (2.2)$$

Here $L(\boldsymbol{E}, \boldsymbol{B})$, which can alternatively be interpreted as a Radon-Nikodym derivative and is therefore often denoted by

$$L = \frac{\mathrm{dP}}{\mathrm{dQ}} \equiv \frac{\mathrm{dP}}{\mathrm{dQ}}(\boldsymbol{E}, \boldsymbol{B}),$$

can be evaluated as follows. The likelihood ratio equals the ratio of the densities of all sampled quantities, where in the numerator these correspond to P and in the denominator to Q. Concretely, with $f_P(\cdot)$ and $f_Q(\cdot)$ the densities of B under P and Q, respectively,

$$L(\boldsymbol{E}, \boldsymbol{B}) = \frac{\lambda e^{-\lambda E_1} f_P(B_1) \cdots \lambda e^{-\lambda E_N} f_P(B_N)}{\lambda_Q e^{-\lambda_Q E_1} f_Q(B_1) \cdots \lambda_Q e^{-\lambda_Q E_N} f_Q(B_N)}.$$

Applying

$$\frac{\lambda}{\lambda_Q} = \frac{1}{\beta(\theta^\star)}, \qquad \frac{f_P(x)}{f_Q(x)} = e^{-\theta^\star x}\beta(\theta^\star),$$

our expression for $L(\boldsymbol{E}, \boldsymbol{B})$ simplifies to

$$\begin{aligned}
\exp\left((\lambda_Q - \lambda)\sum_{i=1}^{N} E_i - \theta^\star \sum_{i=1}^{N} B_i\right) &= \exp\left(-\lambda(1 - \beta(\theta^\star))\sum_{i=1}^{N} E_i - \theta^\star \sum_{i=1}^{N} B_i\right)\\
&= \exp\left(c\,\theta^\star \sum_{i=1}^{N} E_i - \theta^\star \sum_{i=1}^{N} B_i\right)\\
&= e^{-\theta^\star\, X(\tau(u))}.
\end{aligned}$$

From the above, recalling Equation (2.2) as well as the fact that $\tau(u) < \infty$ with probability 1 (under Q, that is), we find the following identity.

Proposition 2.1 *Assume $\theta \in \Theta$. For any $u > 0$,*

$$p(u) = \mathbb{E}_{\mathbb{Q}} e^{-\theta^\star Y(\tau(u))}.$$

Importantly, to obtain Proposition 2.1 no 'asymptotic argumentation' has been applied yet; the identity holds for any $u > 0$. In addition, realizing that (by definition) $Y(\tau(u)) > u$, we have derived an upper bound on $p(u)$, uniform in $u > 0$, known as the *Lundberg inequality*.

Proposition 2.2 *Assume $\theta \in \Theta$. For any $u > 0$,*

$$p(u) \leqslant e^{-\theta^\star u}.$$

▷ *Ladder heights, Cramér-Lundberg approximation.* Observe that we can write $Y(\tau(u)) = u + X(u)$, with $X(u) > 0$ denoting the *overshoot* over level u. Our next objective is to prove that $\mathbb{E}_{\mathbb{Q}} e^{-\theta^\star X(u)}$ tends to a (positive, finite) limit, say γ, as $u \to \infty$, which then implies that

$$\lim_{u \to \infty} p(u)\, e^{\theta^\star u} = \gamma.$$

Let $(H_n)_n$ be the ladder height process corresponding to the net cumulative claim process $Y(t)$, a concept we encountered before in Section 1.4. We refer to Figure 2.3 for an illustration. The individual ladder heights are i.i.d., so that $(H_n)_n$ is a *renewal process* (which is, under $\mathbb{Q}$, non-defective); let H denote a generic ladder height. A crucial observation is that $X(u)$ coincides with the overshoot of $(H_n)_n$ over u. It is a well-known result from renewal theory that, as $u \to \infty$, this overshoot converges (in distribution) to a limiting variable, say $\bar{H}$, with distribution function

$$\mathbb{Q}(\bar{H} \leqslant x) = \int_0^x \frac{\mathbb{Q}(H > y)}{\mathbb{E}_{\mathbb{Q}} H}\,\mathrm{d}y; \tag{2.3}$$

see Exercise A.5. We conclude that

$$\gamma = \mathbb{E}_{\mathbb{Q}}\, e^{-\theta^\star \bar{H}}.$$

We use the theory developed in Section 1.4 to evaluate this object. In particular, the expression that we found for the transform of a ladder height can be applied here.

In order to find an expression for γ, we first determine $\mathbb{E}_{\mathbb{Q}}\, e^{-\alpha H}$. To this end, we can use Proposition 1.4, so as to obtain

$$\mathbb{E}_{\mathbb{Q}}\, e^{-\alpha H} = 1 - \frac{0 - \varphi_{\mathbb{Q}}(\alpha)}{r\theta^\star - r\alpha} = \frac{1}{r}\,\frac{\varphi(\alpha - \theta^\star)}{\alpha - \theta^\star}; \tag{2.4}$$

here it is observed that, in self-evident notation, $\psi_{\mathbb{Q}}(0) = \theta^\star$ (as can be concluded from the right panel of Figure 2.2 in combination with the right panel of Figure 1.3). Then,

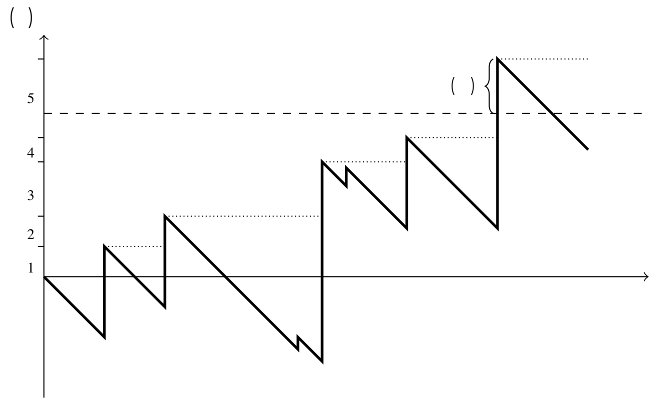

Fig. 2.3 Net cumulative claim process . ¹C°, the ladder height process ¹ =°=, and the overshoot ' ¹D° over level D (dashed line); the corresponding running maximum process .¯ ¹C° is depicted by the dotted lines.

$$\mathrm{E_Q} = \lim_{U\#0} \frac{\mathrm{d}}{\mathrm{d}U}\mathrm{E_Q}\,4^{\;U} = \frac{}{A}\lim_{U\#0}\frac{1 \quad 1^1U \quad \backslash ¢° ¸ ^1U \quad \backslash ¢° 1^0 {}^1U \quad \backslash ¢°}{^1U \quad \backslash ¢°2}$$

$$= \frac{}{A}\frac{1 \quad 1^1 \quad \backslash ¢° \quad \backslash ¢\, 1^0 {}^1 \quad \backslash ¢°}{^1\backslash ¢°2} = \frac{1}{A\backslash ¢}\mathrm{E_Q}.\ ^1 1°-$$

where in the last equality it has been used that \¢ solves i ¹ \¢° = 0, in combination with (2.1). By Equation (2.3),

$$\mathrm{E_Q}\,4^{\;U\,-} = \frac{1}{U}\frac{1}{\mathrm{E_Q}} \quad 1 \quad \mathrm{E_Q}\,4^{\;U} \quad -$$

so that

$$\mathrm{E_Q}\,4^{\;\backslash ¢\,-} = \lim_{U\#\backslash ¢}\frac{1}{U}\frac{1}{\mathrm{E_Q}} \quad 1 \quad \mathrm{E_Q}\,4^{\;U}$$

$$= \frac{1}{\backslash ¢}\frac{1}{\mathrm{E_Q}} \quad 1 ¸ \frac{}{A}1^0{}^1 0° \quad = \quad \frac{1}{A\backslash ¢}\frac{1}{\mathrm{E_Q}}\mathrm{E}.\ ^1 1°-$$

where in the second equality Equation (2.4) has been used. We conclude that W = E. ¹1°•E$_\mathrm{Q}$. ¹1° (which is a positive, finite number). We have thus found the following result.

Theorem 2.1 *Assume 2 . As D ! 1,*

$$?^1D°\,4^{\backslash ¢D} \quad ! \quad \frac{\mathrm{E}.\ ^1 1°}{\mathrm{E_Q}.\ ^1 1°}•$$

We call $p(u) \approx f(u) := \gamma e^{-\theta^* u}$ the *Cramér-Lundberg approximation.* It is asymptotically exact, in that $p(u)/f(u) \to 1$ as $u \to \infty$. Note that we can alternatively write

$$\gamma = \frac{\mathbb{E}X(1)}{\mathbb{E}_{\mathbb{Q}}X(1)} = \frac{\Phi'(0)}{\Phi'_{\mathbb{Q}}(0)} = \frac{r - \mu\,\mathbb{E}B}{\mu_{\mathbb{Q}}\,\mathbb{E}_{\mathbb{Q}}B - r}.$$

⊳ *Large deviations, logarithmic asymptotics.* In case one settles for determining just the exponential decay rate θ^*, a considerably more elementary derivation can be given, using tools from large deviation theory. An additional motivation to include it, is that the same line of reasoning is extensively used in Chapter 8.

A first relevant large deviations result, proven in Exercise 2.4, concerns the probability that the sample mean of a sequence of i.i.d. random variables attains a rare value.

Theorem 2.2 (Cramér) *With $X_1, X_2, \ldots$ i.i.d. random variables distributed as $X(1)$, for $a > \mathbb{E}X(1)$,*

$$\lim_{n\to\infty}\frac{1}{n}\log \mathbb{P}\left(\sum_{i=1}^{n} X_i \geq na\right) = -I(a),$$

where $I(a) := \sup_{\theta>0}(\theta a - \Phi(\theta))$ denotes the Legendre transform of the cumulant generating function $\Phi(\theta)$.

The Legendre transform $I(a)$ featuring in Cramér's theorem is known to be non-negative and convex, and attains its minimal value 0 in $a = \mathbb{E}X(1) = \Phi'(0)$. A second key result from large deviations theory is the following uniform upper bound, which is also proven in Exercise 2.4.

Theorem 2.3 (Chernoff) *Uniformly in n,*

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i \geq na\right) \leq e^{-n I(a)}.$$

A lower bound on the decay rate of $p(u)$ now follows directly. To this end, observe that, for any $t > 0$, we have that $p(u) = \mathbb{P}(\bar{X}(\infty) > u) \geq \mathbb{P}(X(tu) > u)$. As a consequence, for all $t, u > 0$,

$$\frac{1}{u}\log p(u) \geq \frac{t}{tu}\log \mathbb{P}\left(\frac{X(tu)}{tu} > \frac{1}{t}\right).$$

Applying Cramér's theorem, we thus find

$$\liminf_{u\to\infty}\frac{1}{u}\log p(u) \geq -t\, I(1/t).$$

But recall that this lower bound applies to any $t > 0$. Selecting the sharpest among these lower bounds, we arrive at, denoting $\eta^* := t^* I(1/t^*)$ with $t^* := \arg\inf_{t>0} t\, I(1/t)$,

$$\liminf_{u\to\infty}\frac{1}{u}\log p(u) \geq -\eta^*.$$

Below we show that $I^\star = \theta^\star$, but we first prove that the corresponding upper bound, too, equals $I^\star$. The derivation of this upper bound is somewhat more involved. First realize that due to the fact that the process $A(\cdot)$ can go down at a rate of maximally r per time unit, we have

$$p(u) \leq \mathbb{P}\left(\exists n \in \mathbb{N}: \sum_{i=1}^{n} A_i \geq u - r\right) \leq \sum_{n=1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} A_i \geq u - r\right),$$

the second inequality being a consequence of the union bound. For a given $\varepsilon > 0$ (which we will pick later), we can split the expression in the previous display into two sums:

$$\sum_{n=1}^{T^\star(1+\varepsilon)u} \mathbb{P}\left(\sum_{i=1}^{n} A_i \geq u - r\right) + \sum_{n=T^\star(1+\varepsilon)u+1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} A_i \geq u - r\right); \tag{2.5}$$

here we let for ease $T^\star(1+\varepsilon)u$ be integer (remarking that it is straightforward but somewhat annoying to adapt the argument to non-integer values). We consider both sums separately, starting with the second one. This sum is dominated by (letting $u > r$)

$$\sum_{n=T^\star(1+\varepsilon)u+1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{n} A_i \geq 0\right) \leq \sum_{n=T^\star(1+\varepsilon)u+1}^{\infty} e^{-nI(0)} = \frac{e^{-(T^\star(1+\varepsilon)u+1)I(0)}}{1-e^{-I(0)}},$$

where in the first step the Chernoff bound has been used. Now focus on the first sum, which is obviously majorized by

$$T^\star(1+\varepsilon)u \max_{n=1,\ldots,T^\star(1+\varepsilon)u} \mathbb{P}\left(\sum_{i=1}^{n} A_i \geq u - r\right)$$
$$\leq T^\star(1+\varepsilon)u \max_{n=1,\ldots,T^\star(1+\varepsilon)u} \exp\left(-nI\left(\frac{u-r}{n}\right)\right),$$

where in the last step we again appeal to the Chernoff bound. Notice that, by the very definition of $T^\star$, for any $\delta > 0$ and u large enough

$$\exp\left(-nI\left(\frac{u-r}{n}\right)\right) = \exp\left(-(u-r)\frac{n}{u-r}I\left(\frac{u-r}{n}\right)\right) \leq e^{-(u-r)(I^\star-\delta)}$$

for all $n \in \{1,\ldots,T^\star(1+\varepsilon)u\}$. The next step is to pick ε sufficiently large to ensure that $T^\star(1+\varepsilon)I(0) > I^\star - \delta$, entailing that the decay rate of the first sum will dominate. We thus conclude that

$$\limsup_{u\to\infty}\frac{1}{u}\log p(u) \leq \limsup_{u\to\infty}\frac{1}{u}\log\left(T^\star(1+\varepsilon)u\, e^{-(u-r)(I^\star-\delta)}\right) = -I^\star + \delta.$$

Letting $\delta \downarrow 0$ we have established the desired upper bound. Together with the lower bound we have thus derived the following result: the *logarithmic asymptotics* of $p(u)$

are given by

$$\lim_{D\,!\,1} \frac{1}{D} \log ?^{1}D^{o} = \quad ¢•$$

It is left to prove that $¢ = \backslash ¢$. To this end, let $\backslash {}^{1}0^{o}$ be the optimizing argument in the definition of $\;{}^{1}0^{o}$, i.e., $\backslash {}^{1}0^{o}$ solves $0 = \; i^{\,0\,1}\;\backslash^{o}$. Define $\;:= 1•)$ so that $¢ = \inf_{\;i\,0}\;{}^{1}\;{}^{o}•\;$. To find the optimizing , we compute the derivative of ${}^{1}\;{}^{o}•$ and equate it to 0, to obtain the first order condition ${}^{0}1\;{}^{o}\quad{}^{1}\;{}^{o} = 0$, and hence

$$\backslash^{0}1\;{}^{o}\quad{}_{\text{¸}}\;\backslash^{1}\;{}^{o}\;{}_{\text{¸}}\;i^{\,0\,1}\;\;\backslash^{1}\;{}^{oo}\backslash^{0}1\;{}^{o}\qquad{}^{1}\;{}^{o} = 0•$$

But, as we know that ${}_{\text{¸}}\;i^{\,0\,1}\;\;\backslash^{1}\;{}^{oo} = 0$, this condition reduces to $\backslash^{1}\;{}^{o} = \;{}^{1}\;{}^{o}$, i.e., $i^{1}\;\backslash^{1}\;{}^{¢oo} = 0$ for the optimizing $¢$. As a consequence, we have that $\backslash^{1}\;{}^{¢o} = \backslash ¢$. We thus conclude that

$$¢ = \frac{{}^{1}\;{}^{¢o}}{¢} = \frac{¢\,\backslash ¢}{¢} = \backslash ¢•$$

The minimization $\inf_{\;i\,0}\;{}^{1}\;{}^{o}•$ has an appealing interpretation. We can view as the slope at which the process $.\,{}^{1}C^{o}$ moves from level 0 to level D, which 'costs' ${}^{1}\;{}^{o}$ per unit of time. In addition, the time needed to reach level D is proportional to $1•$. Combining both effects we end up with cost ${}^{1}\;{}^{o}•$. When optimizing ${}^{1}\;{}^{o}•$ over , we thus identify the 'cheapest' slope. There is a trade-off: a low value of corresponds to a low cost per unit of time but then the unusual behavior has to persist for a relatively long time, and vice versa for a high value of .

The timescale $)\,¢ := 1•\;¢$ has a similar interpretation: $)\,¢D$ is the typical time it takes to exceed level D. This is also reflected by the fact that the first sum in (2.5), containing timescales around $)\,¢D$, dominates the second sum.

2.3 Subexponential case

We start this section by recalling a result from Chapter 1. We have observed in Proposition 1.1 that we can rewrite

$$?^{1}D^{o} = \mathrm{P}\left(\sum_{8=1}\;\bar{}_{8}\;i\;D\right) = \mathrm{P}\;\;\bar{}^{¢}\;\;i\;D\;\;- \tag{2.6}$$

where the random variable $\bar{}$ is called the 'residual' of , and is geometric with success probability $2 := 1\;\;\_\,\mathrm{E}\;\;•A$. The density of $\bar{}$ is given by $5_{\bar{}}{}^{1}D^{o} := \mathrm{P}^{1}\;\;> D^{o}•\mathrm{E}\;\;$.

In the previous section we assumed that the claim-size distribution was light-tailed, which implied that all moments of the claim-size distribution exist. In this section we consider the case that this condition is violated. Instead we assume that $\bar{}$ is such that, as $D\;!\;1$,

$$\frac{\mathrm{P}^{1}\;\bar{}^{¢2} > D^{o}}{\mathrm{P}^{1}\;\bar{}\; > D^{o}}\;\;!\;\;2•$$

Informally this means that if the sum of two i.i.d. copies of $\bar{B}$ is large, this is most likely due to one of them being large, rather than both of them significantly contributing. We write $\bar{B} \in \mathscr{S}$; the set $\mathscr{S}$ is usually referred to as the set of *subexponential distributions*. It is noted that in general neither $\bar{B} \in \mathscr{S}$ implies $B \in \mathscr{S}$, nor $B \in \mathscr{S}$ implies $\bar{B} \in \mathscr{S}$, in that one can construct counterexamples. These counterexamples are rather intricate, though: as it turns out, for a broad set of relevant distributions, the claims $B \in \mathscr{S}$ and $\bar{B} \in \mathscr{S}$ are equivalent. We get back to this below for three important classes of distributions.

Theorem 2.4 *Assume $\bar{B} \in \mathscr{S}$. As $u \to \infty$,*

$$\frac{p(u)}{\mathbb{P}(\bar{B} > u)} \to \frac{1-\varrho}{\varrho}.$$

Before proving this theorem, we first discuss a number of auxiliary results. These cover a number of useful properties of subexponential distributions. The following lemma is covered by [3, Proposition X.1.7] and [3, Lemma X.1.8]. We do not include the proofs here, but mention that part (i) follows from an inductive argument (noting that the claim is true for $i = 2$ by definition).

Lemma 2.1 *(i) If $X \in \mathscr{S}$, then, as $u \to \infty$,*

$$\frac{\mathbb{P}(X^{*i} > u)}{\mathbb{P}(X > u)} \to i.$$

(ii) If $X \in \mathscr{S}$, then for all $\varepsilon > 0$ there exists a constant C_ε such that, for all i and u,

$$\mathbb{P}(X^{*i} > u) \leqslant C_\varepsilon (1+\varepsilon)^i\, \mathbb{P}(X > u).$$

The following lemma is essentially [3, Lemma X.2.2]. It can be considered as the stochastic counterpart of part (i) of Lemma 2.1, with a random number of terms rather than a deterministic number i.

Lemma 2.2 *Let $X_1, X_2, \ldots$ be i.i.d., distributed as a generic random variable X. Let $N \in \mathbb{N}_0$ be independent of $X_1, X_2, \ldots$ with $\mathbb{E}\, z^N < \infty$ for some $z > 1$. Then, as $u \to \infty$,*

$$\frac{\mathbb{P}(X^{*N} > u)}{\mathbb{P}(X > u)} \to \mathbb{E}\, N.$$

Proof. As $u \to \infty$, by Lemma 2.1(i) and dominated convergence,

$$\frac{\mathbb{P}(X^{*N} > u)}{\mathbb{P}(X > u)} = \sum_{i=0}^{\infty} \frac{\mathbb{P}(X^{*i} > u)}{\mathbb{P}(X > u)}\, \mathbb{P}(N = i) \to \sum_{i=0}^{\infty} i\, \mathbb{P}(N = i) = \mathbb{E}\, N.$$

Here dominated convergence is justified by relying on the following upper bound, that is a consequence of Lemma 2.1(ii):

$$\sum_{i=0}^{\infty} \frac{\mathbb{P}(X^{*i} > u)}{\mathbb{P}(X > u)}\, \mathbb{P}(N = i) \leqslant C_\varepsilon \sum_{i=0}^{\infty} (1+\varepsilon)^i\, \mathbb{P}(N = i) = C_\varepsilon\, \mathbb{E}\, (1+\varepsilon)^N,$$

which is by assumption summable for sufficiently small ε.

The proof of Theorem 2.4 is now straightforward.

Proof (of Theorem 2.4). Combine the geometric sum representation of Equation (2.6) with Lemma 2.2. In addition, observe that

$$\mathbb{E}N = \sum_{i=0}^{\infty} i\,(1-\varrho)^i \varrho = \frac{1-\varrho}{\varrho},$$

and $\mathbb{E}z^N < \infty$ if $z \in (1, 1/(1-\varrho))$.

▷ *Examples.* A few distributions that are known to be subexponential are the Pareto, lognormal, and Weibull distributions. These are characterized by, respectively, for $x > 0$,

$$\mathbb{P}(B > x) = \frac{\kappa^\alpha}{(\kappa + x)^\alpha}, \quad \mathbb{P}(B > x) = 1 - \Phi\left(\frac{\log x - \mu}{\sigma}\right), \quad \mathbb{P}(B > x) = e^{-\mu x^\alpha},$$

with $\Phi(\cdot)$ denoting the distribution function of the standard normal random variable. The following assumptions are imposed on the parameters.

In the Pareto case: $\kappa > 0$ and $\alpha > 1$ (to ensure that $\mathbb{E}B < \infty$).
In the lognormal case: $\mu \in \mathbb{R}$ and $\sigma > 0$.
In the Weibull case: $\mu > 0$ and $\alpha \in (0, 1)$.

It is now a direct consequence of *Pitman's criterion* that each of these random variables B has a subexponential distribution; in Exercise 2.6 Pitman's criterion is formulated and proven. We proceed by arguing that their residuals B^r have subexponential distributions as well, and we in addition identify their tail behavior.

For the Pareto distribution it is directly verified that $\mathbb{E}B = \kappa/(\alpha - 1)$, so that as $x \to \infty$,

$$\mathbb{P}(B^r > x) = \frac{(\alpha-1)}{\kappa}\int_x^\infty \frac{\kappa^\alpha}{(\kappa + y)^\alpha}\,\mathrm{d}y = \frac{\kappa^{\alpha-1}}{(\kappa + x)^{\alpha-1}} \sim \kappa^{\alpha-1} x^{-\alpha+1}. \tag{2.7}$$

Observe that B^r has the same type of tail as B, but with the tail index α being replaced by $\alpha - 1$, which also implies that B^r has a subexponential distribution as well.

For the lognormal case, where $\mathbb{E}B = \exp(\mu + \sigma^2/2)$, more delicate asymptotics are needed. The following lemma is useful in this context. It can be proven by an elementary application of L'Hôpital's rule.

Lemma 2.3 *If for some differentiable function $\gamma(x)$ it holds that*

$$\lim_{x\to\infty} \frac{1}{\gamma(x)}\left(\frac{\gamma(x)}{x}\right)' = 0,$$

then

$$\int_t^\infty \phi(u)e^{-u^2/2}\,du \sim \frac{\phi(t)}{t}e^{-t^2/2}.$$

The first step is to describe the tail behavior of X itself. From its definition, and using the above lemma,

$$\mathbb{P}(X > x) = \int_{(\log x-\mu)/\sigma}^\infty \frac{1}{\sqrt{2\pi}}e^{-y^2/2}\,dy \sim \frac{1}{\sqrt{2\pi}}\frac{\sigma}{\log x-\mu}\exp\left(-\frac{1}{2}\left(\frac{\log x-\mu}{\sigma}\right)^2\right).$$

Now note, in the second equality recalling the expression for $\mathbb{E}X$, that

$$\begin{aligned}\frac{\mathbb{P}(X>x)}{\mathbb{E}X} &= \frac{1}{\mathbb{E}X}\int_x^\infty \frac{1}{\sqrt{2\pi}}\frac{\sigma}{\log y-\mu}\exp\left(-\frac{1}{2}\left(\frac{\log y-\mu}{\sigma}\right)^2\right)dy\\ &= \frac{1}{\mathbb{E}X}\int_{(\log x-\mu)/\sigma}^\infty \frac{1}{\sqrt{2\pi}}\frac{1}{z}e^{-z^2/2}e^{\sigma z+\mu}\,dz\\ &= \frac{\sigma}{\sqrt{2\pi}}\int_{(\log x-\mu)/\sigma}^\infty \frac{1}{z}e^{-(z-\sigma)^2/2}\,dz\\ &= \frac{\sigma}{\sqrt{2\pi}}\int_{(\log x-\mu)/\sigma-\sigma}^\infty \frac{1}{z+\sigma}e^{-z^2/2}\,dz.\end{aligned}$$

Applying Lemma 2.3 once more, we find the following tail for $\bar{X}$:

$$\begin{aligned}\mathbb{P}(\bar{X} > x) &\sim \frac{\sigma}{\sqrt{2\pi}}\frac{\sigma}{\log x-\mu}\frac{\sigma}{\log x-\mu-\sigma^2}\exp\left(-\frac{1}{2}\left(\frac{\log x-\mu}{\sigma}-\sigma\right)^2\right)\\ &\sim \frac{\sigma^3 e^{-\mu-\sigma^2/2}}{\sqrt{2\pi}}\frac{x}{(\log x)^2}\exp\left(-\frac{1}{2}\left(\frac{\log x-\mu}{\sigma}\right)^2\right).\end{aligned}$$

It is again a consequence of Pitman's criterion that the distribution of the residual $\bar{X}$ is subexponential.

We finally consider the Weibull case. A direct calculation shows, with $\Gamma(t)$ denoting the gamma function, that

$$\mathbb{E}X = \left(\frac{1}{\lambda}\right)^{1/\tau}\frac{1}{\tau}\Gamma(1/\tau).$$

Rewriting, with the substitution $\lambda y^\tau = z^2/2$,

$$\int_x^\infty e^{-\lambda y^\tau}\,dy = \frac{2}{\tau}\int_{\sqrt{2\lambda x^\tau}}^\infty (2\lambda)^{-1/\tau} z^{2/\tau-1}e^{-z^2/2}\,dz,$$

a next application of Lemma 2.3 yields

$$\mathbb{P}(\bar{X} > x) \sim \lambda^{1/\tau}x^{1-\tau}\frac{1}{\Gamma(1/\tau)}e^{-\lambda x^\tau}.$$

Again, Pitman's criterion can be used to show that the distribution of ¯ is subexponential.

B *Principle of single big claim.* It can be argued that in the subexponential regime, the event of .¯¹1º exceeding D, for D large, is essentially due to a single big claim. We now present an informal computation to support this claim.

Suppose D is to be exceeded at time C. Then the process .¹Cº is roughly at level $A2C$ = ¹A _ E ºC, so that the big claim arriving at time C should have size at least D ¸ $A2C$. It leads to the approximation, for # 0,

$$?¹Dº \quad \_ \int_0^1 1 \; \mathrm{P}¹ \; ¡ \; D ¸ A2Bº \, \mathrm{d}B•$$

Performing the change of variable E := D ¸ $A2B$, we thus obtain

$$?¹Dº \quad \frac{\_}{A2} \int_D^1 1 \; \mathrm{P}¹ \; ¡ \; Eº \, \mathrm{d}E = \frac{\mathrm{E}}{A2} \mathrm{P}¹ \,¯ > Dº = \frac{1 \quad 2}{2} \mathrm{P}¹ \,¯ > Dº-$$

in line with what was stated in Theorem 2.4.

The above informal procedure can be made rigorous, as pointed out in e.g. [26, pp. 36-39]. The rigorous approach works with a lower bound and an upper bound. In the lower bound, it is used that the ruin probability majorizes the probability of a specific scenario that leads to ruin. Thus, to make the lower bound as tight as possible, one analyzes the probability of the most likely scenario discussed above. In this context, a crucial step concerns the use of the law of large numbers to support the claim that with overwhelming probability .¹Cº is roughly at level $A2C$ at time C, for C sufficiently large. The upper bound is in this type of proofs typically harder to deal with, as one should argue that all other scenarios (e.g. no big claim, multiple big claims) can be asymptotically neglected.

2.4 Time-dependent ruin probability

Thus far, in this chapter we have focused on the asymptotics of the all-time ruin probability ?¹Dº. The analysis of the finite-horizon ruin probability ?¹D-Cº is in general more involved. In this section we provide the main intuition behind the asymptotics of the finite-horizon ruin probability in the regime that the horizon C is scaled by D, i.e., for a given value of C we characterize the behavior of ?¹D-CDº as D ! 1. Again we need to distinguish between the claim sizes being light and heavy tailed.

B *Light-tailed case.* Relying on the large deviations based argumentation introduced in Section 2.2, it directly follows that, for a given value of C ¡ 0,

$$\lim_{D\,!\,1} \frac{1}{D} \log ?¹D-CDº = \inf_{)\,2¹0-C¼}) \quad \frac{1}{)} \;•$$

Calling, for a given C, the corresponding decay rate ¢¹C°, and recalling the definitions of) ¢ and ¢ from Section 2.2, it is immediately seen that ¢¹C° = ¢ in case C >) ¢, whereas ¢¹C° ¡ ¢ if C Ÿ) ¢. The intuition behind this dichotomy is the following. As argued in Section 2.2,) ¢D can informally be seen as the most likely timescale it takes to reach level D. If C >) ¢, then the likelihood of reaching level D before time CD is essentially the same as the likelihood of reaching level D at all, thus explaining that the corresponding decay rates match. If on the contrary C Ÿ) ¢, then it takes additional 'costs' to make sure level D is reached by time CD, so that the likelihood of reaching level D before time CD is significantly lower than the likelihood of reaching level D at all.

B *Subexponential case.* Using the principle of the single big claim, it is straightforward to determine the tail asymptotics of ?¹D-CD°. Indeed, following the reasoning presented in Section 2.3,

?¹D-CD° CD Õ :=0 _ P¹ ¡ D ¸ A2 : ° ! _ 1 CD 0 P¹ ¡ D ¸ A2B° dB-

which, again using the change of variable E := D ¸ A2B, leads to

?¹D-CD° — A2 1 D D¹1¸A2C° P¹ > E° dE

= 1 2 2 P¹ ¯ > D° P¹ ¯ > D¹1 ¸ A2C°° •

This result can be rigorized, e.g., by applying the procedure detailed in [26, pp. 36-39]. For instance in the Pareto case, we thus obtain

?¹D-CD° 1 2 2 D [¸1 1 ¹1 ¸ A2C° [¸1 •

2.5 Heavy tra□c

In this section we consider a second asymptotic regime in which expressions simplify, viz. the regime in which the drift of the net cumulative claim process approaches 0 from above. In other words, we focus on the behavior of ¯¹1° as 2 = 1 _ E •A # 0. In the queueing literature this regime is, for obvious reasons, referred to as the *heavy-tra c regime.* In the insurance risk literature, this is the regime where the *safety loading* A•¹_ E ° 1 is positive but small.

The starting point of our analysis is the Pollaczek-Khinchine formula, as presented in Corollary 1.1:

E 4 U.¯¹1° = A2U / AU _¹1 1¹U°° •

As it turns out, we again have to distinguish between light tails and heavy tails, but, importantly, these notions have a different meaning than in the previous sections: in the light-tailed setting we have that $\mathrm{Var}\, B < \infty$, whereas in the heavy-tailed setting of this section $\mathrm{Var}\, B = \infty$ (where in both cases we assume, as before, that $\mathbb{E}\, B < \infty$). To emphasize its dependence on ε, we write $\bar W_\varepsilon(\infty)$ rather than $\bar W(\infty)$.

▷ *Light-tailed case.* If $\varepsilon \downarrow 0$, then clearly $\bar W_\varepsilon(\infty)$ explodes. We will show, however, that when multiplying this random variable by ε, we have convergence to a non-degenerate limiting random variable. Indeed, using a Taylor series expansion of $b(\varepsilon s)$,

$$\begin{aligned}
\mathbb{E}\, e^{-\varepsilon s \bar W_\varepsilon(\infty)} &= \frac{A\varepsilon^2 s}{A\varepsilon s - (1 - b(\varepsilon s))} \\
&= \frac{A\varepsilon^2 s}{A\varepsilon s - \left(\mathbb{E}\, B\, \varepsilon s - \frac{1}{2}\mathbb{E}[B^2]\,\varepsilon^2 s^2 + o(\varepsilon^2)\right)} \\
&= \frac{A\varepsilon^2 s}{A\varepsilon s - A(1-\varepsilon)\varepsilon s + \frac{1}{2}\mathbb{E}[B^2]\,\varepsilon^2 s^2 + o(\varepsilon^2)} \\
&\to \frac{A}{A + \frac{1}{2}\mathbb{E}[B^2]\, s}.
\end{aligned}$$

as $\varepsilon \downarrow 0$. Note that

$$\lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}[B^2]}{2A} = \theta := \frac{\mathbb{E}[B^2]}{2\,\mathbb{E}\, B}.$$

Hence, by Lévy's convergence theorem (Theorem A.2), and recognizing the Laplace-Stieltjes transform of the exponential random variable, we thus conclude that $\varepsilon \bar W_\varepsilon(\infty)$ converges (as $\varepsilon \downarrow 0$) in distribution to an exponentially distributed random variable with mean θ.

▷ *Heavy-tailed case.* The above reasoning clearly does not apply if $\mathrm{Var}\, B = \infty$, or, equivalently, $\mathbb{E}[B^2] = \infty$. Here we consider the case, for some $\alpha \in (1, 2)$ and $c > 0$,

$$\mathbb{P}(B > x) \sim \frac{c}{\Gamma(1-\alpha)}\, x^{-\alpha}$$

as $x \to \infty$. According to a special case of [7, Theorem 8.1.6], it means that

$$b(s) - 1 + s\,\mathbb{E}\, B \sim c\, s^{\alpha}$$

as $s \downarrow 0$ (for a bit more background on such *Tauberian theorems*, we refer to Exercise 2.7). Now, with $\beta := 1/(\alpha - 1)$, the claim is that $\varepsilon^{\beta} \bar W_\varepsilon(\infty)$ converges to a non-degenerate random variable. This follows by observing that, by expansions similar to those used in the light-tailed setting,

$$\mathbb{E}\, e^{-\varepsilon^{\beta} s \bar W_\varepsilon(\infty)} = \frac{A\varepsilon^{1+\beta} s}{A\varepsilon^{\beta} s - (1 - b(\varepsilon^{\beta} s))} \to \frac{A}{A + c\, s^{\alpha - 1}}. \tag{2.8}$$

as $\alpha \downarrow 0$. We recognize the Laplace-Stieltjes transform of a random variable with a *Mittag-Leffler distribution.*

2.6 Discussion and bibliographical notes

The light-tailed asymptotics, as presented in Section 2.2, have a long history, starting in the 1920s. Lundberg's inequality was given first in [18], whereas the Cramér-Lundberg approximation dates back to [10]. Later efficient proofs relying on change-of-measure have been developed; see e.g. [22]. Our approach also includes some renewal-theoretic elements; see e.g. [2, Chapter 5] for more background. It is also possible to derive the tail asymptotics directly from the Laplace transform of $\bar{\psi}(\cdot)$ through the (not entirely rigorous) Heaviside approach; see e.g. [11, Section 8.1] for a description of this recipe. Arguably the most general version of the Cramér-Lundberg approximation, viz., for spectrally two-sided Lévy processes, can be found in [6]. In the last part of Section 2.2 some elementary results from large deviations theory are relied upon. In the area of large deviations, it is the textbook [12] that perhaps best strikes the balance between completeness, rigor, and accessibility; this book also covers the concepts of change-of-measure and exponential twisting. Our line of reasoning borrows various elements from the argumentation developed in [14].

The tail asymptotics in the heavy-tailed, subexponential case, as presented in Section 2.3, date back to the 1970s. Early references include [5, 8, 19, 24]. In our presentation we essentially followed the line of reasoning of [3, Section X.2]. A general approach has been presented in [13, Section 5], covering besides the light-tailed and heavy-tailed cases also an intermediate case, where $\varphi(\alpha)$ is finite for some positive α but there is no $\theta^\star > 0$ solving $\varphi(\theta^\star) = 0$. For the special case of regularly varying claim sizes an alternative (transform-based) approach can be followed [7]; see also Exercise 2.7. Pitman's criterion can be found in [20]; the line of reasoning in Exercise 2.6 follows that of the proof of [3, Proposition X.1.13]. Asymptotics of $\psi(u) - \psi(u,T)$, for fixed $T > 0$ and $u \to \infty$, assuming regularly varying claim sizes, have been provided in [4]. A textbook treatment of the heavy-tailed case is [17].

In Section 2.4 we presented a brief overview of the results on finite-time ruin probabilities. A very detailed account, including various types of approximations and expansions, is provided in [3, Chapter V]. We in particular mention the classical contributions [1, 21, 23].

Regarding heavy traffic, as discussed in Section 2.5, in the queueing literature the first results on the finite-variance case have appeared in the early 1960s [16], but in the risk setting the first paper goes back to as early as 1940 [15]. A reference on the case with infinite variance is [9]. In both the setting with finite and infinite variance, also process limit versions have been developed. Concretely, after scaling time in an appropriate manner, in the finite-variance case convergence to reflected Brownian motion has been established (of which the exponential distribution is the stationary distribution), whereas in the infinite-variance case there is convergence to a reflected alpha-stable Lévy motion (of which the Mittag-Leffler distribution is the stationary distribution). For a broad overview of such stochastic-process limit results, we refer to the textbook [25].

Exercises

2.1 Consider the case that B is exponentially distributed with parameter μ, and assume $\lambda/(r\mu) < 1$, so that $\bar{Q}(\infty)$ is finite almost surely. Denote $c = 1 - \lambda/(r\mu)$. Recall from part (iv) of Exercise 1.9 that

$$\mathbb{E}\,e^{-\alpha \bar{Q}(\infty)} = \frac{r - \frac{\lambda}{\mu}}{r - \frac{\lambda}{\mu+\alpha}}.$$

(i) Show that $\bar{Q}(\infty)$ is 0 with probability c and exponentially distributed with parameter $\mu - \lambda/r = \mu c$ with probability $1-c$. Show that this entails that, as $x \to \infty$,

$$p(x)\,e^{\mu c x} \to 1 - c.$$

(ii) Verify that the asymptotics of $p(x)$ (as $x \to \infty$), as identified in (i), match with Theorem 2.1. (*Hint*: start by showing that $\theta^* = \mu c$.)

2.2 We again consider the case that B is exponentially distributed with parameter μ, under the assumption $\lambda/(r\mu) < 1$. Denote $c = 1 - \lambda/(r\mu)$.
(i) Show that

$$\psi(a) = \left(\sqrt{\mu(a+r)} - \sqrt{\lambda}\right)^2.$$

(ii) Use this to verify that $\vartheta^* = (1-c)/(rc)$, and to provide the logarithmic asymptotics of $p(x, tx)$ for $x \to \infty$ and any $t > 0$.

2.3 In this exercise we prove Theorem 2.1 using (vi) of Exercise 1.2.
(i) Argue that

$$\lim_{x\to\infty} \mathbb{E}_{\mathbb{Q}}\,e^{-\theta^* O(x)} = \lim_{\alpha\downarrow 0} \frac{\lambda_{\mathbb{Q}}\,\alpha}{\varphi_{\mathbb{Q}}(\alpha)}\,\frac{\beta_{\mathbb{Q}}(\theta^*) - \beta_{\mathbb{Q}}(\psi_{\mathbb{Q}}(0))}{\psi_{\mathbb{Q}}(0) - \theta^*}\,\frac{\beta_{\mathbb{Q}}(\theta^*) - \beta_{\mathbb{Q}}(\alpha)}{\alpha - \theta^*}.$$

(ii) Verify Theorem 2.1 by showing that

$$\lim_{x\to\infty} \mathbb{E}_{\mathbb{Q}}\,e^{-\theta^* O(x)} = \frac{\varphi'(0)}{\varphi'_{\mathbb{Q}}(0)} = \frac{r - \lambda\,\mathbb{E}B}{\lambda_{\mathbb{Q}}\,\mathbb{E}_{\mathbb{Q}}B - r}.$$

2.4 (*) In this exercise we prove the *Chernoff bound* and *Cramér's* theorem. Let $(X_n)_n$ be a sequence of random variables with the cumulant generating function $\log \mathbb{E}\,e^{\theta X_1}$ being finite in an open neighborhood of the origin. Define the Legendre transform of X_1 by, for $a > \mathbb{E}X_1$,

$$I(a) := \sup_{\lambda > 0}(\lambda a - \log \mathrm{E}\, e^{\lambda X_1}).$$

(i) Argue that, for any $\lambda > 0$,

$$\mathrm{P}\left(\sum_{i=1}^{n} X_i \geq na\right) \leq \left(\frac{\mathrm{E}\, e^{\lambda X_1}}{e^{\lambda a}}\right)^{n}.$$

(ii) Show that the result obtained in (i) implies the Chernoff bound

$$\mathrm{P}\left(\sum_{i=1}^{n} X_i \geq na\right) \leq e^{-nI(a)}.$$

(iii) Now that we have verified the Chernoff bound, in order to establish Cramér's theorem, it suffices to prove the lower bound

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathrm{P}\left(\sum_{i=1}^{n} X_i \geq na\right) \geq -I(a). \tag{2.9}$$

With $\lambda = \lambda(a)$ denoting the optimizing argument in the definition of $I(a)$, define the probability measure Q via

$$\mathrm{Q}(X_1 \in \mathrm{d}y) := \mathrm{P}(X_1 \in \mathrm{d}y)\frac{e^{\lambda y}}{\mathrm{E}\, e^{\lambda X_1}},$$

and denote expectations under Q in the usual way. Prove that

$$\mathrm{P}\left(\sum_{i=1}^{n} X_i \geq na\right) = \int \cdots \int_{\{\sum_{i=1}^{n} y_i \geq na\}} \frac{(\mathrm{E}\, e^{\lambda X_1})^n}{e^{\lambda \sum_{i=1}^{n} y_i}} \mathrm{Q}(X_1 \in \mathrm{d}y_1) \cdots \mathrm{Q}(X_n \in \mathrm{d}y_n).$$

(iv) Show that, for all $\varepsilon > 0$,

$$\begin{aligned}\mathrm{P}\left(\sum_{i=1}^{n} X_i \geq na\right) &= (\mathrm{E}\, e^{\lambda X_1})^n\, \mathrm{E}_{\mathrm{Q}}\left(e^{-\lambda \sum_{i=1}^{n} X_i} 1\left\{\sum_{i=1}^{n} X_i \geq na\right\}\right) \\ &\geq (\mathrm{E}\, e^{\lambda X_1})^n e^{-\lambda na(1+\varepsilon)}\, \mathrm{Q}\left(na \leq \sum_{i=1}^{n} X_i \leq na(1+\varepsilon)\right).\end{aligned}$$

(v) Show that $\mathrm{E}_{\mathrm{Q}} X_1 = a$, and verify that $\mathrm{Var}_{\mathrm{Q}} X_1 < \infty$. Then use these results to conclude, by applying the central limit theorem, that, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathrm{Q}\left(na \leq \sum_{i=1}^{n} X_i \leq na(1+\varepsilon)\right) = \frac{1}{2}.$$

(vi) Prove (2.9).

2.5 $(\star)$ This exercise further builds on Exercise 1.5, focusing on the model in which $Y(t)$ is the sum of a compound Poisson process with drift (characterized by the usual λ, $b(\alpha)$, and r) and a Brownian motion. The net cumulative claim process is given by

$$Y(t) := Y_\sigma(t) + \sum_{i=1}^{N(t)} B_i,$$

where $Y_\sigma(t) = -rt + \sigma W(t)$, with $W(t)$ a standard Brownian motion, and r and σ being two positive parameters. In this exercise our objective is to determine the asymptotics (as $u \to \infty$) of $p(u) := \mathbb{P}(\bar{Y}(\infty) > u)$ for $u \in \mathbb{R}_+$. As we have consistently done in this chapter, we impose the requirement $\mathbb{E}\,Y(1) < 0$. Let, as before,

$$\varphi(\alpha) := \log \mathbb{E}\,e^{\alpha Y(1)} = -r\alpha + \frac{\sigma^2\alpha^2}{2} - \lambda(1 - b(\alpha)).$$

(i) As in Exercise 1.5 we work with an auxiliary compound Poisson process $\bar{Y}(t)$. Let ξ_+ and ξ_- be as in Exercise 1.5. Then $\bar{Y}(t)$ is defined by the claim arrival rate $\bar{\lambda} := \xi_-$, claim sizes having the LST

$$\bar{b}(\alpha) := \frac{\xi_+}{\xi_+ + \alpha}\, b(\alpha),$$

and the premium rate being given by $\bar{r} := 1$; let the corresponding Laplace exponent be denoted by $\bar{\varphi}(\alpha) := -\bar{r}\alpha - \bar{\lambda}(1 - \bar{b}(\alpha))$. Let $\theta^\star$ be the solution for θ of $\bar{\varphi}(\theta) = 0$. Show that $\bar{\varphi}(\theta) = 0$ and $\varphi(\theta) = 0$ have the same positive solution (which is $\theta^\star$).
(ii) Let T be an exponentially distributed random variable with parameter μ with $\mu > \theta^\star$, independent of $\bar{Y}(t)$. Prove that, in self-evident notation,

$$\mathbb{P}(T + \bar{Y}(\infty) > u)\, e^{\theta^\star u} \to \frac{\mathbb{E}\,\bar{Y}(1)}{\mathbb{E}_{\mathbb{Q}}\,\bar{Y}(1)}\,\frac{\mu}{\mu - \theta^\star},$$

as $u \to \infty$. (*Hint*: first recall the asymptotics of $\bar{Y}(\infty)$, then condition on the value of T, and use dominated convergence, where a suitable majorant can be found by applying Lundberg's inequality.)
(iii) Recall from Exercise 1.5 that $Y(\infty)$ is distributed as $T + \bar{Y}(\infty)$. Show that this implies that, as $u \to \infty$,

$$\frac{\mathbb{P}(Y(\infty) > u)}{\mathbb{P}(\bar{Y}(\infty) > u)} \to \frac{\xi_+}{\xi_+ - \theta^\star}.$$

Use this result to show that $\mathbb{P}(Y(\infty) > u)\, e^{\theta^\star u}$ converges to a constant as $u \to \infty$.

2.6 $(\star)$ Let X be a non-negative random variable with density $f(\cdot)$. Define the corresponding hazard rate by

$$q(u) := \frac{f(u)}{\mathbb{P}(B > u)}.$$

Our goal is to prove *Pitman's criterion*: B is subexponential if

$$\int_0^\infty e^{u\,q(u)}\, f(u)\,\mathrm{d}u < \infty. \tag{2.10}$$

We do this by showing that, as $u \to \infty$,

$$r(u) := \frac{\mathbb{P}(B^{*2} > u)}{\mathbb{P}(B > u)} - 1 \to 1.$$

(i) Show that

$$r(u) = \frac{\mathbb{P}(B^{*2} > u) - \mathbb{P}(B > u)}{\mathbb{P}(B > u)} = \int_0^u \frac{\mathbb{P}(B > u - v)}{\mathbb{P}(B > u)}\, f(v)\,\mathrm{d}v.$$

(ii) Argue that in order to prove Pitman's criterion we can assume, without any loss of generality, that $q(u)$ is decreasing on $[0,\infty)$.
(iii) With $Q(u) := \int_0^u q(v)\,\mathrm{d}v$, verify that $r(u) := I_1(u) + I_2(u)$, where

$$I_1(u) := \int_0^{u/2} e^{Q(u) - Q(u-v) - Q(v)}\, q(v)\,\mathrm{d}v,$$
$$I_2(u) := \int_0^{u/2} e^{Q(u) - Q(u-v) - Q(v)}\, q(u-v)\,\mathrm{d}v.$$

(iv) Use the fact that $Q(u)$ is concave to show that, for $0 \leqslant v \leqslant u/2$,

$$Q(u) - Q(u - v) \leqslant v\, q(u - v) \leqslant v\, q(v).$$

(v) Apply dominated convergence and the inequalities found in (iv) to prove that, under the proviso that condition (2.10) holds, $I_1(u) \to 1$ and $I_2(u) \to 0$ as $u \to \infty$. Conclude that, under (2.10), $r(u) \to 1$ as $u \to \infty$, as desired.
(vi) Verify for B having a lognormal distribution that B has a subexponential distribution. The same for B having a Weibull distribution (with $\beta \in (0,1)$).

2.7 Consider the situation of subexponential claims, as treated in Section 2.3. For B in special subclasses of $\mathcal{S}$ there are alternative proofs of Theorem 2.4. In this exercise, we make use of the concept of a so-called *Tauberian theorem* to reprove Theorem 2.4 for the case that the claim sizes have a Pareto distribution. Tauberian results provide a one-to-one relation between the behavior of the Laplace-Stieltjes transform of a non-negative random variable Y near the origin and its tail behavior, for the case that the tail distribution of Y is effectively of power-law type (formally: when the tail is *regularly varying*); see e.g. [7, Theorem 1.7.1].

The particular result that we will be using in this exercise states that, for a non-negative random variable Y and some $\alpha \in (1,2)$, the following equivalence holds:

$$\lim_{\alpha\downarrow 0}\frac{\mathbb{E}e^{-\alpha X}-1+\alpha\mathbb{E}X}{\alpha^{\nu}}=\gamma\,\Gamma(-\nu)\iff\lim_{x\to\infty}\mathbb{P}(X>x)\,x^{\nu}=\frac{\gamma}{\Gamma(1-\nu)};$$

here $\gamma > 0$ and $\Gamma(1-\nu) < 0$. Likewise, for a non-negative random variable X and some $\nu\in(0,1)$,

$$\lim_{\alpha\downarrow 0}\frac{\mathbb{E}e^{-\alpha X}-1}{\alpha^{\nu}}=-\gamma\,\Gamma(-\nu)\iff\lim_{x\to\infty}\mathbb{P}(X>x)\,x^{\nu}=\frac{\gamma}{\Gamma(1-\nu)};$$

here $\gamma > 0$ and $\Gamma(1-\nu) > 0$.
(i) Suppose that B has a Pareto distribution: for some $\kappa > 0$ and $\beta\in(1,2)$, the tail distribution of B is given by

$$\mathbb{P}(B>x)=\frac{\kappa^{\beta}}{(\kappa+x)^{\beta}},$$

with $x > 0$. Note that B has a finite mean but an infinite variance. Prove, using the equivalence that was stated above, that

$$\lim_{\alpha\downarrow 0}\frac{b(\alpha)-1+\alpha\mathbb{E}B}{\alpha^{\beta}}=\kappa^{\beta}\,\Gamma(1-\beta)>0.$$

(ii) Show, by combining Corollary 1.1 with Part (i), that

$$\lim_{\alpha\downarrow 0}\frac{\mathbb{E}e^{-\alpha \bar{W}(\infty)}-1}{\alpha^{\beta-1}}=\frac{-}{A2}\,\kappa^{\beta}\,\Gamma(1-\beta).$$

(*Hint*: as a first step show that $(i(\alpha)-A2\alpha)\,\alpha^{-\beta}\to \_\,\kappa^{\beta}\,\Gamma(1-\beta)$ as $\alpha\downarrow 0$.)
(iii) Reprove Theorem 2.4 for this specific class of claim sizes. (*Hint*: use Equation (2.7).)

2.8 Provide the full derivation of (2.8).

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Chapter 3
Regime switching

Abstract — This chapter focuses on the evaluation of the transform of the ruin probability in a regime-switching (or Markov modulated) version of the standard Cramér-Lundberg model. In the derivation various elements from Chapter 1 are relied upon, in particular the idea of conditioning on the first event and the Wiener-Hopf decomposition. The results established in this chapter also facilitate the analysis of the conventional (non-modulated, that is) Cramér-Lundberg model over a phase-type horizon (rather than an exponentially distributed horizon). Finally, we comment on a setup in which the model parameters are periodically resampled, and argue that it fits in the modelling framework discussed in the chapter.

3.1 Introduction

In this chapter we analyze the ruin probability in a model that can be considered as the regime-switching (or Markov modulated) version of the standard Cramér-Lundberg model. Before formally introducing it, we first provide a verbal description. We define a modulating process, in our case a continuous-time Markov chain $X(t)$ on $\{1,\dots,d\}$, also often referred to as the regime process or the background process. In addition, there are d net cumulative claim processes $Y_i(t)$, $i = 1,\dots,d$, all of them corresponding to a compound Poisson process with drift. Then the idea is that the net cumulative claim process $Y(t)$ evolves as the process $Y_i(t)$ when $X(t) = i$.

We proceed by presenting our model in full detail. We start by describing the modulating process. We let $X(t)$ be a Markov process with transition rate matrix Q, that evolves independently of the net cumulative claim processes that we introduce below. Let the jump epochs of this process be $(T_n)_n$. The transition rate matrix $Q = (q_{ij})_{i,j=1}^{d}$ has non-positive entries on the diagonal whereas the other entries are non-negative. Importantly, we do not a priori assume the modulating process to be irreducible; hence, it could correspond to multiple classes. In addition,

$$q_i := -q_{ii} = \sum_{j=1,\, j\neq i}^{d} q_{ij} > 0 \tag{3.1}$$

for all non-absorbing states i, whereas $q_i := -q_{ii} = 0$ for absorbing states i.

The second ingredient of our model are d independent compound Poisson processes with drift, i.e., $Y_1(t), \ldots, Y_d(t)$, evolving independently of the modulating process $X(t)$. In self-evident notation, the Laplace exponent of the i-th process is given by

$$\varphi_i(\alpha) := r_i\alpha - \lambda_i\left(1 - \mathbb{E}\,e^{-\alpha B_i}\right) = r_i\alpha - \lambda_i\left(1 - \mathcal{B}_i(\alpha)\right).$$

Then, in case $X(t) = i$ for $t \in [T_n, T_{n+1})$, the net cumulative claim process $Y(t)$ locally behaves as $Y_i(t)$, in the sense that

$$Y(t) - Y(T_n) = Y_i(t) - Y_i(T_n)$$

for all $t \in [T_n, T_{n+1})$. With $Y_i(0) = 0$ (for all $i = 1, \ldots, d$) this mechanism fully defines the net cumulative claim process $Y(t)$. Note that we do not impose the condition that the premium rates r_i are necessarily positive. We let S be the set of indices i for which $r_i \leq 0$; this is the set of *subordinator states*, i.e., the states i for which $Y_i(t)$ is non-decreasing with probability 1.

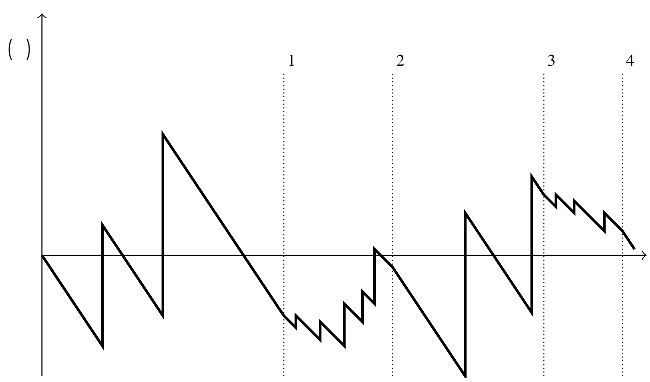

Fig. 3.1 Net cumulative claim process $Y(t)$ for a regime-switching compound Poisson process with $d = 2$. In this example, $X(t) = 1$ for $t \in [0, T_1)$ and $t \in [T_2, T_3)$, whereas $X(t) = 2$ for $t \in [T_1, T_2)$ and $t \in [T_3, T_4)$.

As before, $\bar{Y}(t)$ is denoting the running maximum process of $Y(t)$; likewise, $\bar{Y}_i(t)$ is denoting the running maximum process of $Y_i(t)$, for $i \in \{1, \ldots, d\}$. The primary goal of this section is to evaluate the ruin probability

$$p_i(u,t) := \mathbb{P}\left(\bar{Y}(t) > u \mid X(0) = i\right) = \mathbb{P}_i\left(\bar{Y}(t) > u\right),$$

where $\bar{Y}_i(t)$ equals $\bar{Y}(t)$ conditional on $J(0) = i$. In line with the approach followed in Chapter 1, we settle for a characterization in terms of transforms. More concretely, we will investigate the Laplace transform of $p_i(u,t)$, for $i = 1,\dots,d$, with respect to the initial reserve level u, evaluated at a 'killing epoch' rather than a deterministic epoch. Whereas in Chapter 1 the (exponential) killing rate was constantly β, we now work with a slight generalization: we let the (exponential) killing rate be β_i when the background process is $i \in \{1,\dots,d\}$. We denote the killing epoch by $T_{\boldsymbol{\beta}}$, where the componentwise positive vector $\boldsymbol{\beta} \in \mathbb{R}^d$ equals $(\beta_1,\dots,\beta_d)^\top$; as before, T_β, with a scalar subscript β, still denotes an exponentially distributed random variable with parameter β. In this section we aim to evaluate

$$\pi_i(\alpha,\boldsymbol{\beta}) := \int_0^\infty e^{-\alpha u}\, p_i(u, T_{\boldsymbol{\beta}})\, \mathrm{d}u.$$

This chapter is organized as follows. In Section 3.2 we set up a system of linear equations for the (double) transforms of the probabilities $p_i(u,t)$. This system contains a number of unknown constants, which are identified in Section 3.3. This chapter is concluded, in Section 3.4, by the analysis of the conventional (non-modulated, that is) Cramér-Lundberg model over a phase-type horizon (rather than an exponentially distributed horizon), thus substantially extending the results presented in Chapter 1.

3.2 System of linear equations for transforms

The method we will rely upon is closest to Method 1, as presented in Section 1.3, i.e., conditioning on the first event. Note that there are now *three* types of events, rather than two: as in the conventional Cramér-Lundberg model we have claim arrivals and killing, but now in addition also transitions of the background process $J(t)$. As we will demonstrate below, however, we can conveniently get around the claim arrivals by relying on the Wiener-Hopf decomposition that we discussed in Section 1.3, so that we are left with only killing and the transitions of the background process.

▷ *Non-subordinator case.* The key decomposition that we exploit in our derivations is the following. Suppose $i \in \{1,\dots,d\}\setminus S$, i.e., $Y_i(t)$ is not a non-decreasing process. Given that $J(0) = i$, the time till either killing or a transition of the background process is exponentially distributed with parameter $\nu_i := \beta_i + q_i$. To exceed level u, this can either happen before this epoch, or (in case the event does not correspond to killing) after the background process has jumped to another state. In other words, we can write

$$p_i(u, T_{\boldsymbol{\beta}}) = \mathbb{P}\big(\bar{Y}_i(T_{\nu_i}) > u\big) + \sum_{j\neq i} \frac{q_{ij}}{\nu_i} X_{ij}(u), \tag{3.2}$$

with

$$X_{ij}(u) := \int_0^u \mathbb{P}\big(\bar{Y}_i(T_{\nu_i}) \in \mathrm{d}v,\, Y_i(T_{\nu_i}) + \bar{Y}_j(T_{\boldsymbol{\beta}}) > u\big).$$

where $\bar F_j(\breve T_\#)$ is independent of $(\bar X_i(T_{q_i}), X_i(T_{q_i}))$.

We deal with both terms in the decomposition (3.2) separately, by evaluating their transforms with respect to t. Let us start by considering the first term. The results that were derived in Chapter 1, in particular Equation (1.4), immediately yield that, in self-evident notation,

$$\int_0^\infty e^{-\alpha t}\,\mathbb{P}(\bar X_i(T_{q_i})>t)\,\mathrm{d}t = \frac{1}{\alpha}\left(1-\mathbb{E}\,e^{-\alpha \bar X_i(T_{q_i})}\right) = \frac{1}{\varphi_i(\alpha)-q_i}\left(\frac{\varphi_i(\alpha)}{\alpha}-\frac{q_i}{\psi_i(q_i)}\right).$$

We continue with the second term in (3.2). To this end, first recall that $\bar X_i(T_{q_i})$ and $\bar X_i(T_{q_i}) - X_i(T_{q_i})$ are independent by the Wiener-Hopf decomposition that we established in Section 1.3, with the latter random quantity being exponentially distributed with parameter $\psi_i(q_i)$. Locally writing $\nu_i := \psi_i(q_i)$, it thus follows that

$$\begin{aligned}
&\int_0^\infty e^{-\alpha t}\,x_{ij}(t)\,\mathrm{d}t\\
&= \int_{t=0}^\infty e^{-\alpha t}\int_{s=0}^{t}\int_{r=0}^\infty \mathbb{P}(\bar X_i(T_{q_i})\in \mathrm{d}s)\,\nu_i e^{-\nu_i r} f_j(t-s+r,\breve T_\#)\,\mathrm{d}r\,\mathrm{d}t\\
&= \int_{t=0}^\infty e^{-\alpha t}\int_{s=0}^{t}\int_{u=t-s}^\infty \mathbb{P}(\bar X_i(T_{q_i})\in \mathrm{d}s)\,\nu_i e^{-\nu_i (u-t+s)} f_j(u,\breve T_\#)\,\mathrm{d}u\,\mathrm{d}t,
\end{aligned}$$

where in the last step an elementary change of variables has been performed. Now, interchange the order of the integrals, such that the (easy) integration over t becomes the inner one:

$$\int_{s=0}^\infty\int_{u=0}^\infty\int_{t=s}^{u+s} e^{-(\alpha-\nu_i)t}\,\mathrm{d}t\;\mathbb{P}(\bar X_i(T_{q_i})\in \mathrm{d}s)\,\nu_i e^{-\nu_i (u+s)} f_j(u,\breve T_\#)\,\mathrm{d}u.$$

Evaluating the inner integral and rearranging terms, the expression of the previous display can be rewritten as

$$\frac{\nu_i}{\alpha-\nu_i}\int_0^\infty e^{-\alpha s}\,\mathbb{P}(\bar X_i(T_{q_i})\in \mathrm{d}s)\int_0^\infty \left(e^{-\nu_i u}-e^{-\alpha u}\right) f_j(u,\breve T_\#)\,\mathrm{d}u.$$

Upon combining the above, we have arrived at

$$\int_0^\infty e^{-\alpha t}\,x_{ij}(t)\,\mathrm{d}t = \psi_i(q_i)\,\mathbb{E}\,e^{-\alpha \bar X_i(T_{q_i})}\,\frac{\kappa_j(\psi_i(q_i),\beta)-\kappa_j(\alpha,\beta)}{\alpha-\psi_i(q_i)},$$

which, using the expression derived for the Laplace transform of $\bar X_i(T_{q_i})$, as stated in Theorem 1.1, reduces to

$$q_i\,\frac{\kappa_j(\psi_i(q_i),\beta)-\kappa_j(\alpha,\beta)}{\varphi_i(\alpha)-q_i}.$$

We have established the following result.

Proposition 3.1 *For any $\alpha > 0$ and $\boldsymbol{\beta} \geq \mathbf{0}$, and $i \in \{1,\dots,d\} \setminus S$,*

$$\pi_i(\alpha,\boldsymbol{\beta}) = \frac{1}{\chi_i(\alpha) - \theta_i}\left(\frac{\chi_i(\alpha)}{\alpha} - \frac{\theta_i}{\omega_i(\theta_i)}\right) + \sum_{j \neq i} q_{ij}\,\frac{\pi_j(\omega_i(\theta_i),\boldsymbol{\beta}) - \pi_j(\alpha,\boldsymbol{\beta})}{\chi_i(\alpha) - \theta_i}.$$

▷ *Subordinator case.* Now suppose that $i \in S$: $Y_i(t)$ is a non-decreasing process, i.e., $r_i \geq 0$. Then obviously $\bar{Y}_i(t) = Y_i(t)$ for any $t > 0$. Instead of the decomposition displayed in Equality (3.2), we now have

$$p_i(u, \breve{T}_N) = \mathbb{P}\big(Y_i(T_{\theta_i}) > u\big) + \sum_{j \neq i} \frac{q_{ij}}{\theta_i}\,\eta_{ij}(u), \tag{3.3}$$

with

$$\eta_{ij}(u) := \int_0^u \mathbb{P}\big(Y_i(T_{\theta_i}) \in \mathrm{d}v\big)\,\mathbb{P}\big(\bar{Y}_j(\breve{T}_N) > u - v\big).$$

Concerning the first term in (3.3), we find

$$\int_0^\infty e^{-\alpha u}\,\mathbb{P}\big(Y_i(T_{\theta_i}) > u\big)\,\mathrm{d}u = \frac{1}{\alpha}\left(1 - \mathbb{E}\,e^{-\alpha Y_i(T_{\theta_i})}\right) = \frac{1}{\chi_i(\alpha) - \theta_i}\,\frac{\chi_i(\alpha)}{\alpha}.$$

We continue by computing the Laplace transform of the second term in (3.3). To this end, routine calculations (or the observation that $\eta_{ij}(u)$ is a convolution) yield that

$$\int_0^\infty e^{-\alpha u}\,\eta_{ij}(u)\,\mathrm{d}u = \frac{\theta_i}{\theta_i - \chi_i(\alpha)}\,\pi_j(\alpha,\boldsymbol{\beta}).$$

We arrive at the following result, which aligns with Proposition 3.1 under the choice $\omega_i(\theta_i) = \infty$.

Proposition 3.2 *For any $\alpha > 0$ and $\boldsymbol{\beta} \geq \mathbf{0}$, and $i \in S$,*

$$\pi_i(\alpha,\boldsymbol{\beta}) = \frac{1}{\chi_i(\alpha) - \theta_i}\,\frac{\chi_i(\alpha)}{\alpha} - \sum_{j \neq i} q_{ij}\,\frac{\pi_j(\alpha,\boldsymbol{\beta})}{\chi_i(\alpha) - \theta_i}.$$

▷ *Equations in matrix notation.* Above we found a system of linear equations that defines, for given $\alpha, \boldsymbol{\beta}$, the vector of transforms

$$\boldsymbol{\pi}(\alpha,\boldsymbol{\beta}) = \big(\pi_1(\alpha,\boldsymbol{\beta}),\dots,\pi_d(\alpha,\boldsymbol{\beta})\big)^\top.$$

We now demonstrate that our system of equations can be written in a convenient form using matrix notation. We write

$$\kappa_i(\alpha,\boldsymbol{\beta}) := \frac{\chi_i(\alpha)}{\alpha} - \frac{\theta_i}{\omega_i(\theta_i)}\,1\{i \notin S\} + \sum_{j \neq i} q_{ij}\,\pi_j(\omega_i(\theta_i),\boldsymbol{\beta})\,1\{i \notin S\},$$

with $\kappa(\alpha,\beta)$ the corresponding column vector. In addition, let the (i,j)-th entry of the matrix $M(\alpha,\beta)$ be given by, recalling that $\nu_i = \lambda_i + q_i$,

$$m_{ij}(\alpha,\beta) := \varphi_i(\alpha) - \nu_i\,1\{i=j\} + q_{ij}. \tag{3.4}$$

We end up with the following compact result.

Proposition 3.3 *For any $\alpha > 0$ and $\beta \geqslant \mathbf{0}$,*

$$M(\alpha,\beta)\,\pi(\alpha,\beta) = \kappa(\alpha,\beta).$$

For any given $\alpha > 0$ and $\beta \geqslant \mathbf{0}$ we can thus write $\pi(\alpha,\beta) = M(\alpha,\beta)^{-1}\kappa(\alpha,\beta)$, given the matrix inverse is well-defined. With d denoting the number of states in $\{1,\ldots,d\}\setminus\mathcal{I}$, we observe that, for a given vector β of the killing rates, the characterization of Proposition 3.3 still contains the d unknowns

$$\ell_i(\beta) := \frac{\nu_i}{\psi_i(\nu_i)} + \sum_{j\neq i} q_{ij}\,\varrho_j(\psi_i(\nu_i),\beta)$$

for $i \in \{1,\ldots,d\}\setminus\mathcal{I}$. In Section 3.3 we point out how these constants $\ell_i(\beta)$ can be identified.

We remark that in the usual manner we can convert the vector of transforms of the ruin probabilities $\varrho_i(\alpha,\beta)$ into the vector of the Laplace transforms $\mathbb{E}\,e^{-\alpha \bar{Y}_i(T_\beta)}$ of the running maxima $\bar{Y}_i(T_\beta)$.

3.3 Identification of the unknown constants

The main goal of this section is to identify the unknowns $\ell_i(\beta)$. We do this in three stages: first we consider the case that the state space of $J(t)$ is one recurrent class, then we extend our analysis to the case of one transient class and one recurrent class, and finally we point how to deal with the case of multiple transient classes and one recurrent class.

The following result, which we present without proof, is an immediate consequence of [15, Theorem 1 & Remark 2.1]. It plays a key role in our argumentation.

Proposition 3.4 *Suppose the background process $J(t)$ consists of a single (hence recurrent) class. Let $Y_1(t),\ldots,Y_d(t)$ be compound Poisson processes with (not necessarily negative) drift. Then, for any componentwise positive vector β, the equation $\det M(\alpha,\beta) = 0$ has d solutions for $\alpha \in \mathbb{C}$ that have a positive real part.*

▷ *No transient classes.* We first discuss the case that $\{1,\ldots,d\}$ is a single recurrent class. Define the matrix $M_{i}(\alpha,\beta)$ as the matrix $M(\alpha,\beta)$ but with the i-th column replaced by $\kappa(\alpha,\beta)$. Then, by Proposition 3.3 and Cramer's rule,

$$c_i(\alpha,\beta) = \frac{\det M_{+,i}(\alpha,\beta)}{\det M(\alpha,\beta)}.$$

As $c_i(\alpha,\beta)$ is finite, any zero of the denominator should be a zero of the numerator (in the right half of the complex α-plane, that is). Because in this case the Markov process (J_t) is irreducible, we can apply Proposition 3.4, so as to conclude that $\det M(\alpha,\beta)=0$ has d zeroes in the right half of the complex plane. In the sequel we assume that these zeroes have multiplicity 1; we call them $\alpha_1,\dots,\alpha_d$ - where it should be kept in mind that each of these α_is depends on the vector of killing rates β. In case our multiplicity assumption does not apply, a reasoning similar to the one below still can be used: relying on the concept of Jordan chains, the constants ℓ_i can be identified. We do not provide a detailed discussion of this procedure here, but instead refer to the in-depth treatment in [11].

From the above, we conclude that for a given β and $i=1,\dots,d$ and $j=1,\dots,d$,

$$\det M_{+,i}(\alpha_j,\beta)=0. \tag{3.5}$$

This seemingly yields $d\times d$ equations that we can use to determine the d unknowns $\ell_i(\beta)$ (for $i\in\mathcal{D}$). It can be seen, however, that all equations that correspond to a specific index $j\in\{1,\dots,d\}$ effectively provide the same information. Indeed, with $\boldsymbol{m}_i(\alpha,\beta)$ the i-th column of $M(\alpha,\beta)$, suppose that $\det M(\alpha,\beta)=0$ and that, for a fixed i, $\det M_{+,i}(\alpha,\beta)=0$ for some $\alpha\in\mathbb{C}$ with a positive real part. This means that both $M(\alpha,\beta)$ and $M_{+,i}(\alpha,\beta)$ are singular, and hence there are vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ that are not equal to $\mathbf{0}$ such that

$$\sum_{j=1}^{d}\boldsymbol{m}_j(\alpha,\beta)\,u_j=\mathbf{0},\qquad \sum_{j\neq i}\boldsymbol{m}_j(\alpha,\beta)\,v_j-\boldsymbol{\kappa}(\alpha,\beta)\,v_i=\mathbf{0}.$$

Therefore, for any $i'\neq i$,

$$\begin{aligned}\mathbf{0} &= v_{i'}\sum_{j=1}^{d}\boldsymbol{m}_j(\alpha,\beta)\,u_j-u_{i'}\sum_{j\neq i}\boldsymbol{m}_j(\alpha,\beta)\,v_j-u_{i'}\boldsymbol{\kappa}(\alpha,\beta)\,v_i\\ &= v_{i'}u_i\boldsymbol{m}_i(\alpha,\beta)-\sum_{j\neq i,i'}(u_{i'}v_j-v_{i'}u_j)\,\boldsymbol{m}_j(\alpha,\beta)-u_{i'}v_i\,\boldsymbol{\kappa}(\alpha,\beta),\end{aligned}$$

implying that we have identified a linear combination of the columns of $M_{+,i'}(\alpha,\beta)$ that equals $\mathbf{0}$. This means that $M_{+,i'}(\alpha,\beta)$ is singular, and hence $\det M_{+,i'}(\alpha,\beta)=0$ as well. We thus conclude that, for any given j, varying i in Equation (3.5) does not provide any additional constraints.

Based on the above, for any given index $j=1,\dots,d$ we can thus focus on $\det M_{+,1}(\alpha_j,\beta)=0$ only (we take $i=1$, that is). With $\bar{M}_{ij}(\alpha,\beta)$ representing the $(d-1)\times(d-1)$ matrix which results after deleting the i-th row and the j-th column from $M(\alpha,\beta)$, and recalling that

$$\varphi_i(\alpha,\beta)=\frac{\psi_i(\alpha)}{\alpha}+\ell_i(\beta)1\{i\notin\mathcal{P}\},$$

the equation $\det M_{+,-}(\alpha_j,\beta)=0$ can be rewritten as

$$\sum_{i\in\mathcal{P}}\frac{\psi_i(\alpha_j)}{\alpha_j}(-1)^{1+i}\det M_{i1}(\alpha_j,\beta)+\sum_{i\notin\mathcal{P}}\left(\frac{\psi_i(\alpha_j)}{\alpha_j}+\ell_i(\beta)\right)(-1)^{1+i}\det M_{i1}(\alpha_j,\beta)=0.$$

Through this procedure we obtain d_- equations (i.e., one for each α_j) that are linear in the equally many unknowns $\ell_1(\beta),\ldots,\ell_{d_-}(\beta)$. This system of equations can be dealt with in the standard manner.

◦ *A single transient class.* We now consider the case that there is a single transient class, say $\mathcal{T}\subseteq\{1,\ldots,d\}$, besides the recurrent states; these recurrent states could correspond to a single class or multiple classes.

It is directly seen that, using the procedure we devised for a single recurrent class, we can compute $\pi_i(\alpha,\beta)$ for any recurrent state i, i.e., $i\notin\mathcal{T}$. We therefore focus on $i\in\mathcal{T}$. For these i we write

$$\sum_{j\in\mathcal{T}}m_{ij}(\alpha,\beta)\,\pi_j(\alpha_j,\beta)=\varphi_i(\alpha,\beta)-\sum_{j\notin\mathcal{T}}m_{ij}(\alpha,\beta)\,\pi_j(\alpha,\beta). \tag{3.6}$$

Observe that we already determined the right-hand side. Define by $\bar d:=|\mathcal{T}|$ the number of states in $\mathcal{T}$, and by $\bar d_-:=|\mathcal{T}\cap\mathcal{P}|$ the number of these states that do not correspond to nondecreasing subordinators. In addition, we define the $\bar d\times\bar d$ matrix

$$\bar M(\alpha,\beta):=(m_{ij}(\alpha,\beta))_{i,j\in\mathcal{T}},$$

and we let the $\bar d$-dimensional vector $\bar{\boldsymbol\pi}(\alpha,\beta)$ represent the entries of $\boldsymbol\pi(\alpha,\beta)$ that correspond to the states in $\mathcal{T}$. As a result, we have found the equation, with the vector $\bar{\boldsymbol\kappa}(\alpha,\beta)$ corresponding to the right-hand side of (3.6),

$$\bar M(\alpha,\beta)\,\bar{\boldsymbol\pi}(\alpha,\beta)=\bar{\boldsymbol\kappa}(\alpha,\beta).$$

If we would know that, for any componentwise positive vector $\boldsymbol\beta$, $\det\bar M(\alpha,\beta)=0$ has $\bar d_-$ zeroes in the right half of the complex plane, then we could follow the same approach as the one we developed above for the case with only recurrent states. To this end, in light of Proposition 3.4, we have to verify that the entries of $\bar M(\alpha,\beta)$ can be written in the form (3.4), with transition rates corresponding to a single recurrent class of communicating states. By rewriting the diagonal elements of $\bar M(\alpha,\beta)$ as

$$m_{ii}(\alpha,\beta)=\psi_i(\alpha)-V_i+q_{ii}=\psi_i(\alpha)-\bar V_i+\bar q_{ii}, \tag{3.7}$$

with

$$\bar q_{ii} := \sum_{j\in T\setminus\{i\}} q_{ij}, \quad \bar\beta_i := \left(\beta_i + \sum_{j\in R} q_{ij}\right), \tag{3.8}$$

we conclude that $\bar M(\alpha,\beta)$ has the desired form.

From the above we conclude, by applying Proposition 3.4, that $\bar M(\alpha,\beta) = 0$ indeed has $\bar d$ zeroes in the right half of the complex plane. We can therefore identify the $\rho_i(\beta)$ for $i\in T\setminus S$ (modulo the adaptation that needs to be done in case of roots with multiplicity larger than 1, in line with the remark made above in relation to the case with recurrent states only).

▷ *Multiple transient classes.* We now consider the case that there are K transient classes, say $T_1,\ldots,T_K$ (not to be confused with the exponentially distributed random variable T_β that we have been using elsewhere in this monograph). Let R denote the union of all recurrent states. We write $T_k \to T_{k'}$, with $k,k'\in\{1,\ldots,K\}$, if there is a direct transition from a state in T_k to a state in $T_{k'}$ (i.e., if there is a state $i\in T_k$ and a state $j\in T_{k'}$ such that $q_{ij}>0$). Using this notion, we can order the transient classes in 'layers'. To this end, we use the following recursive definition. Let $L_0 := R$, and

$$L_n := \left\{ T_k : \text{for all } k' \text{ such that } T_k \to T_{k'} \text{ it holds that } k' \in \bigcup_{m=0}^{n-1} L_m \right\}.$$

Observe that the number of layer sets L_n is (including L_0) at most K.

Above we explained how to compute $\varphi_i(\alpha,\beta)$ for $i\in R$ and $i\in L_1$. We now focus on the evaluation of $\varphi_i(\alpha,\beta)$ for $i\in L_n$, having $\varphi_i(\alpha,\beta)$ for $i\in R, L_1,\ldots,L_{n-1}$ at our disposal; evidently, with this evaluation at our disposal we can evaluate all the $\varphi_i(\alpha,\beta)$. Suppose $T_k\in L_n$. As states in L_n have no direct transitions to classes outside L_{n-1}, we have for $i\in T_k$ that

$$\sum_{j\in T_k} m_{ij}(\alpha,\beta)\,\varphi_j(\alpha,\beta) = \psi_i(\alpha,\beta) - \sum_{j\in L_{n-1}} m_{ij}(\alpha,\beta)\,\varphi_j(\alpha,\beta),$$

with the right-hand side containing known quantities only. Now the analysis is as in the case with a single transient class. More specifically, the number of zeroes (in the right half of the complex plane, that is) of the determinant of the matrix $(m_{ij}(\alpha,\beta))_{i,j\in T_k}$ equals the number of states in T_k that do not correspond to non-decreasing subordinators, using the same argument as in the case of a single transient class.

3.4 Cramér-Lundberg model over a phase-type horizon

In Chapter 1 we evaluated, for the conventional Cramér-Lundberg model, the double transform of $p(u,t)$, where the transform over time was interpreted in terms of ruin over an exponentially distributed interval. Interestingly, using the material presented in this chapter, we can immediately extend our findings to ruin over phase-type intervals.

▷ *Phase-type distributions.* The practical relevance of the use of the class $\mathscr{P}$ of phase-type distributions lies in the fact that any distribution on the positive half-line can be approximated arbitrarily closely by a distribution in $\mathscr{P}$ [4, Theorem III.4.2]. It is noted that the proof of this theorem reveals that actually any distribution on the positive half-line can be approximated arbitrarily closely by elements from a class $\mathscr{E} \subset \mathscr{P}$, namely the class of mixtures of Erlang distributions, each of them having the same scale parameter. In this section we show how to evaluate transforms corresponding to a phase-type time interval, thus facilitating the approximation of their counterparts corresponding to a generally distributed time interval.

We proceed by introducing the concept of a phase-type distribution. Such a distribution is characterized by the absorption time of a continuous-time Markov chain. That is, each element in the class of phase-type distributions $\mathscr{P}$ is characterized by (i) a finite state space $\{1,\ldots,d\}$, (ii) an initial probability vector $\delta \in \mathbb{R}^d$, (iii) a $d \times d$ transition rate matrix $Q = (q_{ij})_{i,j=1}^d$ (i.e., it has non-positive diagonal elements, non-negative non-diagonal elements, and row sums equal to zero) and (iv) a non-negative *exit vector* f; for reasons that will become clear below, we use the same symbol d as for the dimension of the background process $J(t)$ in the earlier sections of this chapter. We define the additional transition rate matrix, with $\mathrm{diag}(f)$ a diagonal matrix with f on its diagonal,

$$\bar{Q} := \begin{pmatrix} Q - \mathrm{diag}(f) & f \\ \mathbf{0}^\top & 0 \end{pmatrix}.$$

The dimension of $\bar{Q}$ is $(d+1) \times (d+1)$, where state $d+1$ is usually referred to as the *absorbing state*. Note that $\bar{Q}$ is a genuine transition rate matrix, in that its row sums equal 0. The corresponding phase-type random variable records the time it takes to reach the absorbing state, if the initial state has been drawn according to the distribution δ. We rule out matrices $\bar{Q}$ in which, starting from any state i with $\delta_i > 0$, state $d+1$ is not eventually reached (with probability 1, that is).

▷ *Ruin probabilities over a phase-type horizon.* Now consider a compound Poisson process with negative drift, say $Y(t)$. Let T have a phase-type distribution characterized by the parameters (d, δ, Q, f). The objective is to evaluate

$$\int_0^\infty \int_0^\infty e^{-\alpha x}\, p(x,t)\, \mathbb{P}(T \in dt)\, dx. \tag{3.9}$$

We now point out how the evaluation of this transform falls in the framework of the previous sections. To this end, let $Y_1(t), \ldots, Y_d(t)$ be independent copies of $Y(t)$, such that the compound Poisson process with drift is the same for any state of the background process (say with Laplace exponent $\varphi(\alpha)$). To represent the killed state, let $Y_{d+1}(t) \equiv 0$. Furthermore, we choose $\mathscr{Q} = Q$ and $\beta = f$ such that absorption in state $d+1$ corresponds to killing. It is immediate that the transform in (3.9) equals

$$\sum_{i=1}^{d} x_i\, c_i(\alpha,\beta),$$

with the $c_i(\alpha,\beta)$ that we identified in the earlier sections of this chapter.

▷ *Ruin probabilities over an Erlang horizon.* Above we mentioned that any distribution on the positive half-line can be approximated arbitrarily closely by a distribution in $\mathscr{M}$, i.e., the class of mixtures of Erlang distributions with the same scale parameter. This class is defined as follows. Let $\boldsymbol{\delta} \equiv (x_1,\ldots,x_d)$ be a probability vector, i.e., componentwise non-negative with entries that sum to 1. Let $E_k(\beta)$ be an Erlang distributed random variable with parameters $k \in \mathbb{N}$ and $\beta > 0$, i.e., a random variable whose probability density function is given by

$$\mathbb{P}(E_k(\beta) \in \mathrm{d}t) = e^{-\beta t}\frac{\beta^k t^{k-1}}{(k-1)!}\mathrm{d}t.$$

Then a $T \in \mathscr{M}$, distributed as a mixture of Erlang random variables with the same scale parameter, is characterized by the vector $\boldsymbol{\delta}$ as introduced above, $\beta > 0$, and $\boldsymbol{k} \in \mathbb{N}^d$. Its probability density function is given by

$$\mathbb{P}(T \in \mathrm{d}t) = \sum_{i=1}^{d} x_i\, \mathbb{P}(E_{k_i}(\beta) \in \mathrm{d}t).$$

Hence, in order to evaluate Expression (3.9) for $T \in \mathscr{M}$, it suffices to be able to evaluate it for an $E_k(\beta)$-distributed horizon. Indeed, if we can compute

$$\begin{aligned} c^{(k)}(\alpha,\beta) &:= \int_0^\infty\int_0^\infty e^{-\alpha u}\, p(u,t)\, \mathbb{P}(E_k(\beta)\in \mathrm{d}t)\,\mathrm{d}u \\ &= \int_0^\infty\int_0^\infty e^{-\alpha u}\, p(u,t)\, e^{-\beta t}\frac{\beta^k t^{k-1}}{(k-1)!}\mathrm{d}t\,\mathrm{d}u, \end{aligned} \tag{3.10}$$

then (3.9) can be expressed as

$$\sum_{i=1}^{d} x_i\, c^{(k_i)}(\alpha,\beta).$$

As we point out now, $c^{(k)}(\alpha,\beta)$ can be computed in a straightforward manner from $c(\alpha,\beta) \equiv c^{(1)}(\alpha,\beta)$, which is the transform corresponding to the ruin probability over an exponentially distributed horizon. To this end we define

$$c'(\alpha,\beta) := \frac{\mathrm{d}}{\mathrm{d}\beta}c(\alpha,\beta).$$

Proposition 3.5 *For $k \in \mathbb{N}$,*

$$c^{[k]}(\alpha,\beta) = \sum_{\ell=0}^{k-1} \frac{(-\beta)^\ell}{\ell!}\, c^{(\ell)}(\alpha,\beta). \tag{3.11}$$

Proof. We use that (3.10) implies that

$$c^{[k]}(\alpha,\beta) = \frac{(-\beta)^k}{(k-1)!}\,\frac{\mathrm{d}^{k-1}}{\mathrm{d}\beta^{k-1}}\frac{c(\alpha,\beta)}{\beta}.$$

Observing that, by the binomium,

$$\frac{\mathrm{d}^{k-1}}{\mathrm{d}\beta^{k-1}}\frac{c(\alpha,\beta)}{\beta} = \sum_{\ell=0}^{k-1}\binom{k-1}{\ell}\, c^{(\ell)}(\alpha,\beta)\,\frac{(k-1-\ell)!}{(-\beta)^{k-\ell}}, \tag{3.12}$$

the stated follows immediately.

3.5 Resampling

In the models that are typically considered in the ruin theory literature the underlying model primitives are fixed. In the context of the conventional Cramér-Lundberg model that was introduced in Chapter 1, this concretely means that the claim arrival rate, the premium rate, and the claim-size distribution are held constant, in the sense that they do not change over time. In reality, however, such a setup is typically not valid: as a consequence of various 'external circumstances' the model primitives may fluctuate. One could think of exogenous factors affecting the claim arrival process, such as the state of the economy, the political situation, weather conditions, and policy regulations. Neglecting the parameter uncertainty (by using the conventional Cramér-Lundberg model with time-averaged parameters) could evidently lead to a substantial underestimation of the ruin probability.

To remedy the above shortcoming, in the present section the following model is considered. Let $X_i(t)$, for $i = 1,\dots,d$ be independent compound Poisson processes (with drift, that is), characterized by their respective Laplace exponents $\varphi_1(\cdot)$ up to $\varphi_d(\cdot)$. As before, we have

$$\varphi_i(\alpha) = r_i\alpha - \lambda_i(1 - b_i(\alpha)),$$

i.e., $X_i(t)$ corresponds to a claim arrival rate λ_i, a premium rate r_i, and the claim-size distribution having Laplace-Stieltjes transform $b_i(\alpha)$. Let $(T_n)_n$ be the jump epochs of a Poisson process with rate $\nu > 0$; we set $T_0 := 0$. At these Poisson instants with probability $p_i \in [0,1]$ the process $X_i(t)$ is picked (independently of anything that happened in the past), with the p_i summing to 1; this $X_i(t)$ is 'active' until the next Poisson instant. It means that we can recursively define the net cumulative claim process in the following manner. Considering $t \in [T_n, T_{n+1})$, and supposing that for these t the process $X_i(t)$ was picked, the net cumulative claim process satisfies

$$Y(t) - Y(T_n) = Y_i(t) - Y_i(T_n),$$

Observe that this construction is such that in the resulting model the parameters of the driving compound Poisson process are periodically (i.e., at Poisson instants) resampled according to the probabilities p_i. As a consequence, with probability p_i the net cumulative claim process $Y(t)$ locally behaves as $Y_i(t)$ (being parameterized by λ_i, μ_i, and $b_i(\alpha)$).

We proceed by arguing that the model proposed above actually fits in the framework of the model introduced in Section 3.1. To this end, let us construct the associated transition rate matrix Q, that should be such that the resulting modulating process $X(t)$ corresponds to the resampling mechanism described above. Considering the transition rates from state i, we observe that the process $X(t)$ stays for an exponentially distributed amount of time (with rate, say, ν_i) in i; after this time, it jumps to state $j \neq i$ with probability $p_j/(1-p_i)$. The parameter ν_i can be determined by computing the Laplace-Stieltjes transform of the time spent in state i, say τ_i. It readily follows that

$$\mathbb{E}\, e^{-\alpha \tau_i} = \sum_{k=1}^{\infty} p_i^{k-1}(1-p_i)\left(\frac{\nu}{\nu+\alpha}\right)^k = \frac{(1-p_i)\nu}{\alpha+(1-p_i)\nu},$$

which reveals that the sojourn time τ_i has an exponential distribution with parameter $\nu_i = (1-p_i)\nu$. We observe that we should pick

$$q_{ij} = \frac{\nu_i p_j}{1-p_i} = \nu p_j$$

for $i \neq j$, whereas $q_{ii} = -\nu_i$. We thus arrive at

$$Q = \nu \mathbf{1}\boldsymbol{p}^{\top} - \nu I_d, \tag{3.13}$$

with $\mathbf{1}$ an all-ones vector and I_d the d-dimensional identity matrix. The conclusion is that our process $Y(t)$ fits in the model introduced in Section 3.1, and hence all results from Sections 3.2 and 3.3 apply.

3.6 Discussion and bibliographical notes

We end this chapter by briefly discussing relevant references. An early source on processes under Markov modulation is [9], covering the broader class of *Markov additive processes*; see also the textbook account in [4, Chapter XI].

There is a broad array of papers that characterize the ruin probability under regime switching, or, essentially equivalently, the queue's workload distribution under Markov modulation. In this context we start by remarking that ruin-theoretic results under regime switching allow a direct translation in terms of their queueing-theoretic counterparts; for an in-depth treatment of this issue we refer to [14]. We now provide a series of key references on the evaluation of the ruin probability under regime switching. The case in which all processes $._1(\cdot),\ldots,._3(\cdot)$ are deterministic drifts is covered, in the queueing context, by e.g. [3, 17, 18]; these systems are often referred to as *Markov fluids*. The case of Lévy processes with one-sided jumps is addressed in e.g. [6, 11, 14]. Whereas the above references impose the assumption that the background process is irreducible, the approach followed in the present chapter, which is based on [19], applies to any chain structure; see also the precursor paper [13], which works with a specific non-irreducible background process.

Proposition 3.4 has been established in [15]. Results on maxima over phase-type and Erlang horizons, related to those presented in Section 3.4, can be found in [1], but also in [12, Section IV.1], [19, Section 5], and [21]. Proposition 3.5 was taken from [8, Section 5].

The resampling model discussed in Section 3.5 has been proposed in [10]. As pointed out in detail in [10], the fact that the matrix $"(U)$ (for the $\&$ matrix constructed in Section 3.5) is the sum of a diagonal matrix and a rank-one matrix, as we have seen in (3.13), implies that the corresponding eigenvalues can be characterized relatively explicitly (and, in particular, all of them are real). We refer to [2, 16] for related papers, the crucial difference with [10] being that in [2, 16] the arrival rate is random but sampled just once, whereas in [10] in principle the full Laplace exponent of the cumulative claim process is random and resampled on a periodic basis. Related resampling mechanisms are studied in [7, 20]; in some of them the resampling takes place from a distribution with uncountable support.

Exercises

3.1 Consider the case $3 = 2$, with none of the two driving compound Poisson processes with drift (i.e., the processes $._1(\cdot)$ and $._2(\cdot)$) being a non-decreasing subordinator, and the chain structure of the modulating process being defined by

$$\& = \begin{pmatrix} @ & @ \\ 0 & 0 \end{pmatrix},$$

for some $@ > 0$. The goal of this exercise is to compute the transforms $c_1(U,\beta)$ and $c_2(U,\beta)$.

(i) Show that $c(U,\beta)$ can be written as

$$\left(\frac{i_1(U)/U - l_1(\beta)}{i_1(U) \;\; V_1 \;\; @}\quad \frac{@\,(i_2(U)/U - l_2(\beta))}{(i_1(U) \;\; V_1 \;\; @)(i_2(U) \;\; V_2)}, \frac{i_2(U)/U - l_2(\beta)}{i_2(U) \;\; V_2}\right)^{>},$$

for constants $l_1(\beta)$ and $l_2(\beta)$.

(ii) Show that

$$l_2(\beta) = \frac{V_2}{k_2(V_2)}$$

and, with $b := k_1(V_1 + @)$,

$$I_1(\beta) = \frac{V_1 + @}{k_1(\backslash_1)} + \frac{@}{i_2(b) \quad V_2} \frac{i_2(b)}{b} \frac{V_2}{k_2(V_2)} .$$

(iii) Conclude that

$$c_2(U, \beta) = \frac{1}{i_2(U) \quad V_2} \frac{i_2(U)}{U} \frac{V_2}{k_2(V_2)} ,$$

which is in line with (1.4), and

$$c_1(U, \beta) = \frac{1}{i_1(U) \quad V_1 \quad @} \frac{i_1(U)}{U} \frac{V_1 + @}{b} + @ \frac{c_2(b, \beta) \quad c_2(U, \beta)}{i_1(U) \quad V_1 \quad @} .$$

3.2 (¢) Again consider the case $3 = 2$, but now $._2(C)$, one of the two driving compound Poisson processes with drift, is a non-decreasing subordinator. Let

$$\& = \begin{matrix} @_1 & @_1 \\ @_2 & @_2 \end{matrix} ,$$

for some $@_1, @_2 \; ¡ \; 0$, so that the modulating process consists of a single class. We focus on the analysis of the probability of eventual ruin, in that we take the killing rate equal to 0 (i.e., $V_1 = V_2 = 0$). In this exercise, we develop two ways to evaluate $c_8(U)$, the transforms of the all-time ruin probabilities $?_8(D)$, for $8 = 1, 2.$ The first approach relies on the techniques developed in the present chapter, whereas the second approach directly exploits ideas from Chapter 1. We throughout assume

$$\frac{@_2}{@_1 + @_2} i_1^0(0) + \frac{@_1}{@_1 + @_2} i_2^0(0) \; ¡ \; 0, \tag{3.14}$$

to make sure that $\mathbb{E}\, .(1) \; \ddot{Y} \; 0$, and hence that $.(C)$ does not grow to 1 almost surely as $C \; ! \; 1$. We also define the matrix

$$''(U) := \begin{matrix} i_1(U) \quad @_1 & @_1 \\ @_2 & i_2(U) \quad @_2 \end{matrix} .$$

Observe that necessarily $i_1(U) > 0$ and $i_2(U) \; 6 \; 0$ for all $U > 0.$

(i) Define $\;(U) := \det \; ''(U) = i_1(U)\, i_2(U) \quad @_1\, i_2(U) \quad @_2\, i_1(U)$. Using the results from the present chapter, verify that $\pi(U)$ equals

$$\frac{1}{\;(U)} \begin{matrix} i_2(U) \quad @_2 & @_1 \\ @_2 & i_1(U) \quad @_1 \end{matrix} \begin{matrix} i_1(U).U + I_1 \\ i_2(U).U \end{matrix} ,$$

for a constant I_1.

(ii) By virtue of [15, Theorem 2], under the condition (3.14), there is a single U with a non-negative real part such that $\;(U) = 0$, which we call U_0. Show that this U_0 equals 0 and

$$I_1 = \frac{@_1}{i_2{}^1U_0{}^o \quad @_2} \quad \frac{i_2{}^1U_0{}^o}{U_0} \quad \frac{i_1{}^1U_0{}^o}{U_0} = \frac{@_1}{@_2} \quad i_2^{0\,1}0^o \quad i_1^{0\,1}0^o \bullet$$

Evaluate $c_1{}^1U^o$ and $c_2{}^1U^o$.
(iii) We now proceed with the alternative approach. Our first focus is on evaluating the objects, for $8 = 1\text{-}2$,

$$r_8{}^1U^o := \mathrm{E}\ 4^{\ U.\bar{}\ ^1 1^o}\ \mathrm{j}\ \ {}^1 0^o = 8\ \bullet$$

To compute the $r_8{}^1U^o$, we define an alternative compound Poisson process $.\ \ {}^1C^o$. It is characterized by the claim arrival rate $\_\ \ := k_1{}^1@_1{}^o$, claim sizes having the LST

$$1\ \ {}^1U^o := \mathrm{E}\,4^{\ U.\bar{}_1{}^1)_{@_1}{}^o}\,\mathrm{E}\,4^{\ U.{}_2{}^1)_{@_2}{}^o} = \frac{U \quad k_1{}^1@_1{}^o}{i_1{}^1U^o \quad @_1}\,\frac{@_1}{k_1{}^1@_1{}^o} \quad \frac{@_2}{@_2 \quad i_2{}^1U^o}\text{-}$$

and the premium rate being given by $A\ \ := 1$; let the corresponding Laplace exponent be denoted by $i\ \ {}^1U^o := A\ U\ \ \_\ {}^1 1\ \ 1\ {}^1U^{oo}\bullet$ Show that

$$r_1{}^1U^o = \mathrm{E}\,4^{\ U.\bar{}_1{}^1)_{@_1}{}^o}\ r\ \ {}^1U^o\text{-}$$
$$r_2{}^1U^o = \mathrm{E}\,4^{\ U.\bar{}_1{}^1)_{@_1}{}^o}\,\mathrm{E}\,4^{\ U.{}_2{}^1)_{@_2}{}^o}\ r\ \ {}^1U^o = 1\ \ {}^1U^o\ r\ \ {}^1U^o\text{-}$$

where

$$r\ \ {}^1U^o = \frac{U\,{}^1 i\ \ {}^{o0}1 0^o}{i\ \ {}^1U^o}\bullet$$

(*Hint*: for $8 = 2$ use Figure 3.2, while for $8 = 1$ a similar argument can be used.)
(iv) Translate the formulas for the $r_8{}^1U^o$ into their counterparts for the $c_8{}^1U^o$, for $8 = 1\text{-}2$. Check that these agree with the answers found under (ii).

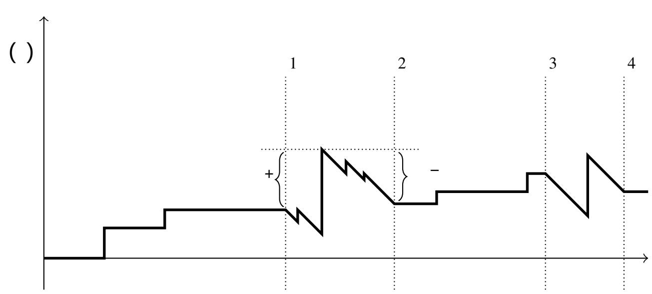

Fig. 3.2 Net cumulative claim process $.\ \ {}^1C^o$ for the regime-switching compound Poisson process of Exercise 3.2. In this example, ${}^1 0^o = 2$; more precisely, ${}^1C^o = 2$ for $C\ 2\ »0\text{-}\ *_1{}^o$ and $C\ 2\ »*_2\text{-}\ *_3{}^o$, whereas ${}^1C^o = 1$ for $C\ 2\ »*_1\text{-}\ *_2{}^o$ and $C\ 2\ »*_3\text{-}\ *_4{}^o$. Observe that $/\ ›$ is distributed as $\bar{.}_1{}^1)_{@_1}{}^o$. Also, $/$ is distributed as $\_\,{}_1{}^1)_{@_1}{}^o$, which is exponential with parameter $k_1{}^1@_1{}^o$, and independent of $/\ ›$.

3.3 In Section 3.4 we have presented a method to compute the transform of the ruin probability over an Erlang horizon. In this exercise we develop an alternative approach. Recall that $T_k(\beta)$ is an Erlang distributed random variable with parameters $k \in \mathbb{N}$ and $\beta > 0$, and that $c_k(\alpha,\beta)$ is the transform of the ruin probability over an $T_k(\beta)$ horizon, as defined through (3.10). As before, $\varphi(\cdot)$ is the Laplace exponent of the process $Y(t)$, while $\psi(\cdot)$ is its right inverse.

(i) Argue that

$$p(u, T_k(\beta)) = \int_{x=u}^{\infty} \mathbb{P}(\bar{Y}(T_\beta) \in dx) + \int_{x=0}^{u}\int_{y=0}^{\infty} \mathbb{P}(\bar{Y}(T_\beta) \in dx)\, \psi(\beta)\, e^{-\psi(\beta) y}\, p(u-x+y, T_{k-1}(\beta))\, dy.$$

(ii) Prove that

$$\int_{u=0}^{\infty} e^{-\alpha u} \int_{x=u}^{\infty} \mathbb{P}(\bar{Y}(T_\beta) \in dx)\, du = \frac{1 - r(\alpha,\beta)}{\alpha} = c(\alpha,\beta),$$

with $r(\alpha,\beta)$ as introduced in Section 1.2, and $c(\alpha,\beta) = c_1(\alpha,\beta)$.

(iii) Prove that

$$\begin{aligned}
&\int_{u=0}^{\infty} e^{-\alpha u} \int_{x=0}^{u}\int_{y=0}^{\infty} \mathbb{P}(\bar{Y}(T_\beta) \in dx)\, \psi(\beta)\, e^{-\psi(\beta) y}\, p(u-x+y, T_{k-1}(\beta))\, dy \\
&\quad = \psi(\beta)\, r(\alpha,\beta)\, \frac{c_{k-1}(\psi(\beta),\beta) - c_{k-1}(\alpha,\beta)}{\alpha - \psi(\beta)} \\
&\quad = \beta\, \frac{c_{k-1}(\psi(\beta),\beta) - c_{k-1}(\alpha,\beta)}{\varphi(\alpha) - \beta},
\end{aligned}$$

so that

$$c_k(\alpha,\beta) = c(\alpha,\beta) + \beta\, \frac{c_{k-1}(\psi(\beta),\beta) - c_{k-1}(\alpha,\beta)}{\varphi(\alpha) - \beta}.$$

(iv) Now the $c_k(\alpha,\beta)$ can be determined recursively. We demonstrate this procedure by computing $c_2(\alpha,\beta)$. Verify that

$$c(\psi(\beta),\beta) = c_1(\psi(\beta),\beta) = \frac{1 - \beta\psi'(\beta)}{(\psi(\beta))^2}.$$

Use this to evaluate $c_2(\alpha,\beta)$.

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Chapter 4
Interest and two-sided jumps

Abstract — In this chapter we present an in-depth analysis of a generalization of the Cramér-Lundberg model. Three distinguishing additional elements have been incorporated: (i) the insurance firm receives interest over its surplus level, (ii) besides claims, leading to negative jumps of the surplus level process, we also allow positive jumps, and (iii) as before we obtain the probability of ruin (transformed with respect to the initial capital surplus level) before an exponentially distributed time, but now jointly with three other quantities: the corresponding time of ruin, the undershoot, and the overshoot.

4.1 Introduction

One could argue that the conventional Cramér-Lundberg model (discussed in detail in Chapter 1) lacks realism, in that it ignores various relevant features. A second obvious shortcoming is that Cramér-Lundberg theory primarily focuses on the analysis of ruin probabilities, while in a practical context various other metrics are of crucial interest as well, such as the (negative) value of the surplus level process immediately after ruin (sometimes referred to as the *loss*, or *ruin deficit*).

This chapter intends to remedy the above issues by substantially generalizing the results of Chapter 1. More concretely, compared to the conventional Cramér-Lundberg analysis, three additional elements have been incorporated:

The insurance firm receives interest over its surplus level. We apply an interest rate $A > 0$.

Besides claims, leading to negative jumps of the surplus level process, we also allow positive jumps (which could be thought of as capital injections).

As before we aim at characterizing the probability of ruin (transformed with respect to the initial capital surplus level) before an exponentially distributed time, but now jointly with three other quantities [7, 8]: the corresponding time of ruin, the undershoot, and the overshoot.

This chapter is organized as follows. In Section 4.2 we formally introduce our model, and present some preliminaries. Then, in Section 4.3 we consider the case in which the upward jumps are exponentially distributed. Section 4.4 explains how the model's underlying exponentiality assumptions can be relaxed, which in particular allows the upward jumps to be substantially more general than exponentially distributed.

4.2 Model and notation

As in Chapter 1, our surplus process is denoted by $X_u(t)$, with $X_u(0) = u$ being the initial surplus level. Define the ruin time by

$$\tau(u) := \inf\{t > 0 : X_u(t) < 0\},$$

so that the probability of loss before or at time t equals $\mathbb{P}(\tau(u) \leq t)$. Whereas in Chapter 1 it was our objective to analyze the double transform

$$\varphi(\alpha, \beta) := \int_0^\infty \int_0^\infty \beta e^{-\alpha u - \beta t} p(u, t) \, du \, dt,$$

we now focus on a more general object, also including the time of ruin $\tau(u)$, the undershoot $X_u(\tau(u)-)$, and the overshoot $X_u(\tau(u))$. As mentioned, the overshoot can be seen as the insurance company's loss (i.e., the amount of capital lost upon ruin).

Concretely, we wish to compute, for a given initial capital surplus level u,

$$p(u, t, \gamma) := \mathbb{E}\, e^{-\gamma_1 \tau(u) - \gamma_2 X_u(\tau(u)-) - \gamma_3 X_u(\tau(u))} 1\{\tau(u) \leq t\},$$

where $\gamma \equiv (\gamma_1, \gamma_2, \gamma_3)^\top$. As the computation of this object is not within reach, we resort to evaluating its transform. With T_β as before an exponentially distributed time with parameter β sampled independently from everything else, and again taking the transform with respect to u, we consider

$$\varphi(\alpha, \beta, \gamma) := \int_0^\infty \int_0^\infty e^{-\alpha u} \beta e^{-\beta t} p(u, t, \gamma) \, dt \, du = \int_0^\infty e^{-\alpha u} p(u, T_\beta, \gamma) \, du.$$

For conciseness, in the sequel we consistently write, for given $\beta > 0$ and γ such that $\gamma_1, \gamma_2 \geq 0$ and $\gamma_3 \leq 0$ (where we recall that $\tau(u)$ and $X_u(\tau(u)-)$ are non-negative, and $X_u(\tau(u))$ is non-positive),

$$p(u) \equiv p(u, T_\beta, \gamma) = \mathbb{E}\, e^{-\gamma_1 \tau(u) - \gamma_2 X_u(\tau(u)-) - \gamma_3 X_u(\tau(u))} 1\{\tau(u) \leq T_\beta\}.$$

The model we consider in this chapter is more realistic than the conventional Cramér-Lundberg model, because of the following elements that have been incorporated:

As before we have claims arriving according to a Poisson process. These correspond to *downward* jumps in the surplus level process $X_u(t)$. We let $\lambda_- > 0$ be the corresponding arrival rate, and $b_-(\alpha)$ the LST of the generic (non-negative) claim size B_- (where it is assumed that the claim sizes constitute a sequence of i.i.d. random variables, which is independent of the arrival epochs). In addition, however, we allow independent *upward* jumps in $X_u(t)$, which could for instance represent capital injections. These arrive, independently of the claim arrival process, according to a Poisson process with rate $\lambda_+ > 0$. We let $b_+(\alpha)$ be the LST of the generic upward jump B_+, where it is assumed that the upward jumps form a sequence of i.i.d. random variables (independent of the corresponding arrival process).

In the second place, we let the insurance company receive *interest* (at rate $r > 0$) over its current surplus level. With T_i denoting the i-th jump epoch of the surplus level process $X_u(t)$, this means that, between two consecutive jump epochs T_i and T_{i+1}, the surplus process evolves according to the ordinary differential equation

$$\mathrm{d}X_u(t) = r\,\mathrm{d}t + r\,X_u(t)\,\mathrm{d}t.$$

It is readily verified that, for $t \in (T_i, T_{i+1})$,

$$X_u(t) = X_u(T_i)\,e^{r(t-T_i)} + \frac{r}{r}\left(e^{r(t-T_i)} - 1\right), \tag{4.1}$$

where the obvious limit is taken in the case of zero interest (i.e., $r = 0$), so that then $X_u(t)$ equals $X_u(T_i) + r(t - T_i)$. Figure 4.1 presents a sample path of the surplus level process until ruin.

We recover the results dealt with in Chapter 1 when plugging in $\gamma = 0$ into $c(\alpha, \beta, \gamma)$, and in addition assuming (i) that there are no upward jumps (i.e., $\lambda_+ = 0$), and (ii) that no interest is earned over the insurance firm's surplus level (i.e., $r = 0$).

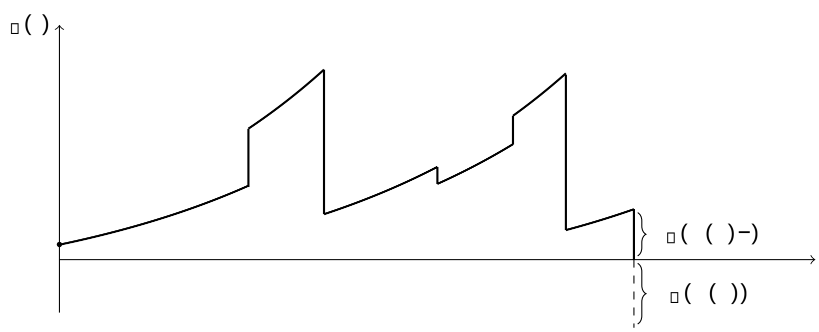

Fig. 4.1 Sample path of $X_u(t)$ until $\tau(u)$. Upward jumps are distributed as the generic random variable B_+, downward jumps are distributed as the generic random variable B_-.

4.3 Exponential upward jumps

In this section we focus on the case that the upward jumps B_+ are exponentially distributed with parameter $\mu > 0$. Later in this chapter we consider the extension to a setup in which the upward jumps stem from a broad class of phase-type distributions. The approach that we follow resembles the one used in Exercise 1.2: an integro-differential equation for $p(u)$ is first derived, then this is transformed with respect to u, and finally the resulting ordinary differential equation is solved.

▷ *Setting up an integro-differential equation for $p(u)$.* The first step in our analysis concerns the characterization of $p(u)$ through an integro-differential equation. We do so relying on classical 'Markovian reasoning', borrowing elements from both Method 1 (see Section 1.3) and Method 4 (see Section 1.6) — a similar approach has been followed in Exercise 1.2. With the total arrival rate being given by $\lambda := \lambda_- + \lambda_+$, and using that the surplus level behaves according to (4.1) between consecutive jumps, as $\Delta t \downarrow 0$,

$$\begin{aligned} p(u) = e^{-\alpha_1 \Delta t}\Big(&\lambda_- \Delta t \int_0^u \mathbb{P}(B_- \in dv)\, p(u-v) \\ &+ \lambda_- \Delta t \int_u^\infty \mathbb{P}(B_- \in dv)\, e^{-\alpha_2 u} e^{-\alpha_3 (v-u)} \\ &+ \lambda_+ \Delta t \int_0^\infty \mu e^{-\mu v} p(u+v)\, dv \\ &+ (1 - \lambda \Delta t - V \Delta t)\, p(u + r\Delta t + r u \Delta t) + o(\Delta t)\Big). \end{aligned} \tag{4.2}$$

This relation can be understood as follows. In the first place, observe that in the considered time interval of length Δt the time till ruin $\tau(u)$ grows by Δt. In the second place, the undershoot $X_u(\tau(u)-)$ and the overshoot $-X_u(\tau(u))$ can be assigned their values when the surplus level drops below 0, which can only happen due to a negative jump of size at least u.

By linearizing $e^{-\alpha_1 \Delta t}$ as well as $p(u + r\Delta t + r u \Delta t)$, we can rewrite the relation (4.2) to

$$\begin{aligned} p(u) = p(u + r\Delta t + r u\Delta t) &+ \lambda_- \Delta t \int_0^u \mathbb{P}(B_- \in dv)\, p(u-v) \\ &+ \lambda_- \Delta t \int_u^\infty \mathbb{P}(B_- \in dv)\, e^{-\alpha_2 u} e^{-\alpha_3 (v-u)} \\ &+ \lambda_+ \Delta t \int_0^\infty \mu e^{-\mu v} p(u+v)\, dv - (\alpha_1 + \lambda + V)\, \Delta t\, p(u) + o(\Delta t), \end{aligned}$$

as $\Delta t \downarrow 0$. As we wish to derive an integro-differential equation, we subtract the term $p(u + r\Delta t + r u \Delta t)$ from both sides. Subsequently dividing by Δt, we thus obtain

$$\frac{?¹D \, ¸ A \; C ¸ A \; D \; C^º \quad ?¹D^º}{A \; C ¸ A \; D \; C} \, ¹A ¸ A \; D^º = \_ \int_0^{D} P¹ \quad 2\,\mathrm{d}E^º \, ?¹D \quad E^º$$

$$¸ \_ \int_{1}^{1} P¹ \quad 2\,\mathrm{d}E^º \, 4^{\;W_2 D} \, 4^{\;W_3 ¹D \;\; E^º}$$

$$¸ \_ ¸ \int_0^{1^D 1} \text{`}4^{\;\text{`}E} \, ?¹D \, ¸ \, E^º \, \mathrm{d}E \quad ¹W_1 \, ¸ \_ \, ¸ \, V^º \, ?¹D^º \, ¸ \, >¹1^º-$$

as $C \# 0$. As a last step we take the limit $C \# 0$, so as to arrive at the following integro-differential equation.

Lemma 4.1 *For any* D ¡ 0,

$$?^0¹D^º \, ¹A ¸ A \; D^º = \_ \int_{0}^{D} P¹ \quad 2\,\mathrm{d}E^º \, ?¹D \quad E^º$$

$$¸ \_ \int_{1}^{1} P¹ \quad 2\,\mathrm{d}E^º \, 4^{\;W_2 D} \, 4^{\;W_3 ¹D \;\; E^º}$$

$$¸ \_ ¸ \int_0^{1^D 1} \text{`}4^{\;\text{`}E} \, ?¹D \, ¸ \, E^º \, \mathrm{d}E \quad ¹W_1 \, ¸ \_ \, ¸ \, V^º \, ?¹D^º•$$

B *Setting up a differential equation for* $\bar{c}¹U^º$. With Lemma 4.1 at our disposal, the next goal is to evaluate $\bar{c}¹U^º \quad \bar{c}¹U- V- \gamma^º := Uc¹U- V- \gamma^º$, which can be interpreted as $?¹D^º$ in which the initial surplus level D is exponentially distributed with parameter U• This concretely means that we transform the full integro-differential equation of Lemma 4.1 with respect to D. To this end, multiply both sides by $U4^{\;\;UD}$, and integrate over $D \; 2 \; ¹0- \; 1^º$, with the objective to obtain an equation that is fully expressed in terms of the (yet unknown) function $\bar{c}¹U^º$. We do so by considering each of the individual terms separately.

We start by computing the transforms of the terms on the left-hand side of the equation. By applying integration by parts it is seen that

$$\int_0^{1} ?^0¹D^º \, A \, U4^{\;\;UD} \, \mathrm{d}D = AU \;\; ?¹0^º \quad \bar{c}¹U^º \;\; ;$$

cf. Appendix A.2.
Similarly,

$$\int_0^{1} ?^0¹D^º \, A \;\; D \, U4^{\;\;UD} \, \mathrm{d}D = A \;\; U \int_0^{1} ?¹D^º \;\; 4^{\;\;UD} \quad D \, U4^{\;\;UD} \;\; \mathrm{d}D = A \;\; U\bar{c}^0¹U^º-$$

where we have made use of the standard identity

$$\bar{c}^0¹U^º = \frac{\bar{c}¹U^º}{U} \int_0^{1} D \, U4^{\;\;UD} \, ?¹D^º \, \mathrm{d}D•$$

We continue by considering the terms on the right-hand side of the equation. The first term yields, upon interchanging the order of the integrals,

$$
\_ \int_0^{1}\int_{1}^{1} {}^{1}_{0}\,{}^{D}_{1}\; P^1 \quad 2\,\mathrm{d}E^{o}\, ?^1D \quad E^{o}\,\mathrm{d}E \quad U4^{\;UD}\,\mathrm{d}D
$$

$$
= \_ \int_0^{1} 4^{\;UE} \int_E^{1} ?^1D \quad E^{o}\, U4^{\;U^1D\;\;E^{o}}\,\mathrm{d}D \quad P^1 \quad 2\,\mathrm{d}E^{o} = \_\; 1\;\; {}^1U^{o}\bar{c}^1U^{o}\bullet
$$

The second term gives (cf. a discussion in Appendix A.3):

$$
\_ \int_0^{1}\int_D^{1} P^1 \quad 2\,\mathrm{d}E^{o} 4^{\;W_2D}\, 4^{\;W_3{}^1D\;\;E^{o}} \quad U4^{\;UD}\,\mathrm{d}D
$$

$$
= \_\; U \int_0^{1} \frac{4^{W_3E} \quad 4^{\;{}^1U\,\text{¸}\,W_2{}^{o}E}}{U\ \text{¸}\ W_2\ \text{¸}\ W_3} P^1 \quad 2\,\mathrm{d}E^{o} = \_\; U\, \frac{1\ \ {}^1\ \ W_3{}^{o} \quad 1\ \ {}^1U\ \text{¸}\ W_2{}^{o}}{U\ \text{¸}\ W_2\ \text{¸}\ W_3}\bullet
$$

Here it is noticed that $U = \quad W_2 \quad W_3$ is a removable singularity.

Applying the transformation $F := D\ \text{¸}\ E$ (and adapting the integration area in the standard way), the third term in the right-hand side reads

$$
\_\text{¸} \int_0^{1}\int_0^{1} \text{`e}^{\;\text{`}E}\, ?^1D\ \text{¸}\ E^{o}\,\mathrm{d}E \quad U4^{\;UD}\,\mathrm{d}D
$$

$$
= \_\text{¸}\,\frac{\text{`}}{\text{`}\quad U}\bar{c}^1U^{o} \quad \_\text{¸}\,\frac{U}{\text{`}\quad U}\bar{c}^{1\,\text{`}\,o}\bullet \tag{4.3}
$$

Notice that here $U = \text{`}$ corresponds to a removable singularity. The above expression is defined for all $U > 0$, but $U = \text{`}$ will require some extra care in the calculations that follow.

And finally, by the definition of $\bar{c}^1U^{o}$,

$$
\int_0^{1} {}^1W_1\ \text{¸}\ \_\ \text{¸}\ V^{o}\, ?^1D^{o}\, U4^{\;UD}\,\mathrm{d}D = \quad {}^1W_1\ \text{¸}\ \_\ \text{¸}\ V^{o}\,\bar{c}^1U^{o}\bullet
$$

The next step is to write the resulting identity in a compact form. To this end we introduce some notation: we let $\quad := \ {}^1W_1\ \text{¸}\ V^{o}\bullet A$ and

$$
{}^1U^{o} := \ {}^{-1}U^{o}\ \text{¸}\ \frac{}{U}\text{-} \qquad {}^{-1}U^{o} := \frac{A}{A} \quad \frac{\_}{A}\,\frac{1 \quad 1\ \ {}^1U^{o}}{U}\ \text{¸}\ \frac{\_\text{¸}}{A}\,\frac{1}{\text{`}\quad U}\text{-}
$$

$$
{}^1U^{o} := \frac{\_}{A}\,\frac{1\ \ {}^1\ \ W_3{}^{o} \quad 1\ \ {}^1U\ \text{¸}\ W_2{}^{o}}{U\ \text{¸}\ W_2\ \text{¸}\ W_3} \quad \frac{A}{A}\,?^10^{o} \quad \frac{\_\text{¸}}{A}\,\frac{1}{\text{`}\quad U}\bar{c}^{1\,\text{`}\,o}\bullet
$$

Then it is readily checked that we have derived the following inhomogeneous ordinary differential equation for $\bar{c}^{1\ o}$.

Proposition 4.1 *For any $U > 0$, $\bar{c}^{1\ o}$ fulfils the di erential equation*

$$
\bar{c}^{0\,1}U^{o} = \quad {}^1U^{o}\,\bar{c}^1U^{o}\ \text{¸}\ \quad {}^1U^{o}\bullet \tag{4.4}
$$

Remark 4.1 Above we tacitly assumed a positive interest rate: $A \ \ ¡\ 0$. For $A\ = 0$ the differential equation (4.4) turns into an algebraic equation. It is directly seen that then $c^1U^{o} = \bar{c}^1U^{o}\bullet U$ equals $\#^1U^{o}\bullet \ \ {}^1U^{o}$, with

$$\#¹U° := \; \_ \; \frac{1 \;¹\; W_3° \quad 1 \;¹U ¸ W_2°}{U ¸ W_2 ¸ W_3} \; ¸ A?¹0° \; ¸ \_ ¸ \grave{} \; \frac{1}{U} \bar{c}¹ \grave{} °-$$

$$¹U° := AU \quad \_ \; ¹1 \quad 1 \; ¹U°° \; ¸ \_ ¸ \grave{} \; \frac{U}{U} \quad W_1 \quad V•$$

This expression is a true extension of (1.4), in that plugging in $W_1 = W_2 = W_3 = \_ ¸ = 0$ indeed yields (1.4); the constant $?¹0°$ follows from the usual argument that nonnegative zeroes of the denominator should be zeroes of the numerator as well. Clearly, we get more detailed information now: besides the probability of ruin, in addition the time to ruin, the undershoot, and the overshoot (cf. Exercise 1.2). ^

B *Solving the di erential equation for* $\bar{c}¹U°$. We continue by examining the first-order inhomogeneous differential equation (4.4), assuming that $A \; ¡ \; 0$ (the case $A = 0$ is discussed in Remark 4.4). It is routinely solved using the method of variation of constants; however, as we shall see below, the behavior of $\bar{c}¹U°$ at $U = \grave{}$ requires some attention. We find, with $¢¹U°$ the primitive of $¹U°$:

$$\bar{c}¹U° = \int_0^U ¹[° \exp \; ¢¹[° \; d[\; ¸ \quad \exp \; ¢¹U° \; • \tag{4.5}$$

As a consequence of the fact that $¢¹U° \; ! \; 1$ as $U \; ! \; 1$ (where one should bear in mind that $¹U° \; ! \; A•A$ as $U \; ! \; 1$), we have that $\bar{c}¹1° = ?¹0° \; 2 \; ¹0\text{-}1°$ necessarily implies that

$$= \int_0^1 ¹[° \exp \; ¢¹[° \; d[• \tag{4.6}$$

This implies that

$$\bar{c}¹U° = \int_U^1 ¹[° \exp \; ¢¹[° \; d[\; \exp \; ¢¹U° \; • \tag{4.7}$$

Observe that we are done, up to the determination of the two unknown constants $?¹0°$ and $\bar{c}¹\grave{}°$. The next step is to identify constraints by which these constants can be determined. To this end, we write $¹U° = ?¹0° \; {}_1¹U° \; ¸ \; \bar{c}¹\grave{}° \; {}_2¹U° \; ¸ \; {}_3¹U°$- where the functions ${}_:¹U°$ (with $: = 1\text{-}2\text{-}3$) are given by

$${}_1¹U° := \frac{A}{A}\text{-} \qquad {}_2¹U° := \frac{\_¸}{A} \frac{1}{\grave{} \quad U}\text{-} \qquad {}_3¹U° := \frac{\_}{A} \frac{1 \;¹\; W_3° \quad 1 \;¹U ¸ W_2°}{U ¸ W_2 ¸ W_3}•$$

Analogously, we define $¹U°$ as $?¹0° \; {}_1¹U° \; ¸ \; \bar{c}¹\grave{}° \; {}_2¹U° \; ¸ \; {}_3¹U°$- where

$${}_:¹U° := \int_U^1 {}_:¹[° \exp \; ¢¹[° \; d[\text{-}$$

for $: = 1\text{-}2\text{-}3$. To obtain constraints that can be used to determine $?¹0°$ and $\bar{c}¹\grave{}°$, we consider the singularities around 0 and $\grave{}$ in greater detail. Here the main underlying idea is that, obviously, if for some U we have that $¢¹U° = 1$, then necessarily $¹U° = 0$, as a consequence of (4.7) and the finiteness of $\bar{c}¹U°$.

The shape of $(\,)$ reveals that, for some constant $_0 < 0$,

$$\lim_{u\downarrow 0} \frac{\phi(u)}{\log u} = {}_0,$$

which implies that $\phi(u) \to \infty$ as $u \downarrow 0$. This means that to make sure (4.7) holds, we must have that $(0) = 0$ (so that $= 0$). Using this relation, we can now express the constants $p(0)$ and $\bar{c}(`)$ in one another. As (u) is linear in these two constants, we find

$$p(0)\,{}_1(0) + \bar{c}(`)\,{}_2(0) = {}_3(0). \tag{4.8}$$

Analogously, for some constants $\bar{\ } ` \in \mathbb{R}$ and $` < 0$,

$$\lim_{u\uparrow `} \frac{\phi(u) - \bar{\ } `}{\log(` - u)} = `,$$

entailing that $\phi(u) \to \infty$ as $u \uparrow `$. Relation (4.7) thus gives that $(`) = 0$, which in turn leads to the relation

$$p(0)\,{}_1(`) + \bar{c}(`)\,{}_2(`) = {}_3(`). \tag{4.9}$$

We have thus found the two linear equations (4.8) and (4.9), in equally many unknowns. It takes an elementary calculation to verify that

$$p(0) = \frac{{}_3(0)\,{}_2(`) - {}_3(`)\,{}_2(0)}{{}_1(0)\,{}_2(`) - {}_1(`)\,{}_2(0)}, \quad \bar{c}(`) = \frac{{}_1(0)\,{}_3(`) - {}_1(`)\,{}_3(0)}{{}_1(0)\,{}_2(`) - {}_1(`)\,{}_2(0)}. \tag{4.10}$$

We have thus arrived at the following result, providing a full characterization of $\bar{c}(u)$.

Theorem 4.1 *If $A > 0$, then $\bar{c}(u)$ equals (4.7), with $p(0)$ and $\bar{c}(`)$ given by (4.10).*

Remark 4.2 Consider the integrals appearing in (4.8) and (4.9), which are the coefficients and right-hand side of the derived two-dimensional system of linear equations. In the above argumentation, we tacitly assumed that these integrals are well-defined. This remark serves to provide a justification for this claim. Importantly, this property is not evident, because of the u and $` - u$ appearing in the denominators of the functions involved.

First note that $\phi(u)$ behaves as uA/A for large u. Recalling that, for any $f \in \mathbb{R}$, $g > 0$ and > 0,

$$\int_g^\infty x^f e^{-x}\,dx < \infty,$$

we observe that to establish finiteness of the numerator and denominator, we only have to deal with their singularities at $u = 0$ and $u = `$.

We proceed by studying the integrals appearing in (0). Recall that $u = w_2 - w_3$ is a removable singularity (appearing in $_3(u)$). For $[\downarrow 0$ and $: = 1, 2, 3$, $_:([)$

is a bounded function (in that it is finite on the interval ¹0-Y° for some small Y). Observe that we can write ¢¹[° = j₀ log [¸ ǰ₀¹[° for some ǰ₀¹[° that is bounded on ¹0-Y° and j₀ ¡ 0• It thus follows from

$$\int_0^{Y} [^{j_0} \mathrm{d}[\ddot{Y} 1$$

that, for : = 1-2-3,

$$\int_0^{Y} :^1[^o \exp \quad ¢^1[^o \, \mathrm{d}[\ddot{Y} 1•$$

Having settled the 'well-definedness' around 0, we continue by pointing out how to deal with the singularity of ₂¹U° at U = `. For [ " ` we can write ₂¹[° = b`•¹` [° ¸ b̄`¹[° for a bounded function b̄`¹[° (i.e., on the interval ¹` Y- `°) and some b`; also, ¢¹[° = j` log¹` [° ¸ ǰ`¹[° for ǰ`¹[° bounded on ¹0-Y° and j` ¡ 0• Because

$$\int_{`\,Y}^{`} {}^1` \; [^o \; {}^{1,j`} \, \mathrm{d}[\ddot{Y} 1$$

it follows that

$$\int_{`\,Y}^{`} {}_2{}^1[^o \exp \quad ¢^1[^o \, \mathrm{d}[\ddot{Y} 1•$$

Clearly, the integral involving ₂¹[° between ` and ` ¸ Y can be dealt with analogously. Also, by the boundedness of ₁¹U° and ₃¹U° around `, the integrals involving ₁¹U° and ₃¹U° do not lead to any complications at `. Combining the above, this proves the finiteness of the integrals in (4.8). The three integrals appearing in (4.9) can be dealt with in an analogous way. ^

Remark 4.3 It is noted that c̄¹0° can be found using L'Hôpital's rule:

$$\bar{c}^1 0^o = \lim_{U \# 0} \frac{{}^1U^o}{4 \;\; ¢^1U^o} = \lim_{U \# 0} \frac{{}^1U^o}{{}^1U^o} = 0•$$

This could also be concluded via another argumentation:

$$\bar{c}^1 0^o = \lim_{D\,!\,1} ?^1 D^o \; 6 \lim_{D\,!\,1} \mathrm{P}^1 g^1 D^o \; 6 \;)_V \, j \; -_D{}^1 0^o = D^o = 0•$$

In addition, along similar lines one can verify that one indeed obtains c̄¹`° when taking the limit U ! `. ^

B *Alternative description of c̄¹U°.* We continue by developing an alternative way to describe c̄¹ °, namely through a power series expansion. Writing, for coefficients 5̄ and 6 ,

$$\bar{\ }^1U^o = \sum_{=0}^{\infty} \bar{5} \; U \; - \qquad {}^1U^o = \sum_{=0}^{\infty} 6 \; U \; -$$

we have found the differential equation

$$\bar c'(U) = \left(\sum_{k=0}^{\infty} \bar f_k U^k + \frac{}{U}\right)\bar c(U) + \sum_{k=0}^{\infty} g_k U^k.$$

Writing $c_k := \bar c^{(k)}(0)$, this differential equation can be rewritten to

$$\sum_{k=0}^{\infty} \frac{c_{k+1}}{k!}U^k = \left(\sum_{k=0}^{\infty} \bar f_k U^k + \frac{}{U}\right)\sum_{k=0}^{\infty}\frac{c_k}{k!}U^k + \sum_{k=0}^{\infty} g_k U^k.$$

Notice that this relation implies that $c_0 = \bar c(0) = 0$, in line with what we concluded in Remark 4.3. It is further observed that, by collecting the terms corresponding to the same power in both sides of the equation, the coefficients c_k can be determined. After some algebra, we find that the c_k obey the following recursion.

Proposition 4.2 *The power series expansion of $\bar c(U)$ is $\sum_{k=0}^{\infty} c_k U^k/k!$, where $c_0 = 0$ and, for $k \in \mathbb{N}$,*

$$c_{k+1} = \frac{1}{k!}\,\frac{}{(1 + k)!}\left(\sum_{m=0}^{k-1} \bar f_m c_{k-m} + g_k\right).$$

Remark 4.4 In case $A = 0$, the argumentation has to be set up slightly differently, as we do not have $\varphi(U) \to \infty$ as $U \to \infty$, which means that we cannot conclude that equals the right-hand side of (4.6). At the same time, note that in this case no constant $p(0)$ plays a role (as follows directly from the definition of (U)). The singularities at 0 and , however, still exist. In this remark we explain how linear equations can be found that determine the unknown constants and $\bar c()$.

Write, for $k = 2,3$,

$$_k(U) := \int_0^U {}_k(y)\exp(\varphi(y))\,\mathrm{d}y,$$

and $(U) = \bar c()\,_2() + {}_3(U)$. Using the same argumentation as before, and recalling (4.5), for any value of U such that $\varphi(U) = \infty$, we necessarily have that $(U) + = 0.$ This means that we have thus found two linear equations, namely

$$\bar c()\,_2(0) + {}_3(0) + = \bar c()\,_2() + {}_3() + = 0.$$

Using $_2(0) = {}_3(0) = 0$, we conclude

$$\bar c() = -\frac{_3()}{_2()}, \qquad = 0.$$

Mimicking the line of reasoning of Remark 4.2, it can be shown that all integrals involved are well-defined. ◇

4.4 Relaxation of the exponentiality assumptions

One of the seeming drawbacks of the results presented so far concerns the exponentiality assumptions that we imposed. Concretely, $\bar{c}^1U^o$ corresponds to the situation in which the initial surplus level, the killing, and the upward jumps stem from exponential distributions. In this section we discuss how the exponentiality assumptions can be greatly relaxed.

B *Initial level and time horizon.* As pointed out in Section 3.4, we can approximate any distribution on the positive half-line by a distribution in the class of phase-type distributions • Actually a smaller class of distributions suffices, viz. , the class of mixtures of Erlang distributions. In this context it is important to notice that also any deterministic number can be approximated arbitrarily closely by an element from this class; indeed, as can be verified using the law of large numbers, an Erlang distribution with shape parameter : and scale parameter :•I (for : 2 N and I ¡ 0) converges to the deterministic number I as : ! 1. Below we explain how to compute ?1*-)-γo with a random initial level * and a random time horizon), both in the class . This clearly extends the results of Section 4.3 considerably, as there the focus was on the evaluation of $\bar{c}^1U^o$ = ?1*U-)V-γo, with *U an exponentially distributed random variable with parameter U (and)V an exponentially distributed random variable with parameter V, as before).

As argued in Section 3.4, so as to be able to deal with distributions in , it suffices to deal with * and) being Erlang distributed. This means that we can once more rely on Proposition 3.5, to translate results for * or) being exponentially distributed to their Erlang counterpart. The following example presents an explicit procedure for doing so, dealing with an Erlang distributed initial surplus level *.

Example 4.1 Consider the context of Theorem 4.1, but now we want the initial level to be sampled from an Erlang distribution with parameters : and U (i.e., with mean :•U), rather than from the exponential distribution. Proposition 3.5 requires the evaluation of the derivatives $\bar{c}^{1\ o}$1 o• We know $\bar{c}^1U^o$, so (4.4) provides us with

$$\bar{c}^{1 1^o}1U^o = \quad 1U^o\bar{c}^1U^o \ \text{¸} \quad 1U^o•$$

But then also

$$\begin{aligned}\bar{c}^{1 2^o}1U^o &= \quad {}^{1 1^o}1U^o\bar{c}^1U^o \ \text{¸} \quad 1U^o\bar{c}^{1 1^o}1U^o \ \text{¸} \quad {}^{1 1^o}1U^o\\ &= \quad {}^{1 1^o}1U^o \ \text{¸}\ 1 \quad 1U^{oo2} \ \ \bar{c}^1U^o \ \text{¸} \quad 1U^o \quad 1U^o \ \text{¸} \quad {}^{1 1^o}1U^o•\end{aligned}$$

Continuing along these lines, we can compute $\bar{c}^{1\ o}1U^o$ recursively in terms of $\bar{c}^1U^o$. More precisely, it can be checked that $\bar{c}^{1\ o}1U^o$ = 1U^o $\bar{c}^1U^o$ ¸ 1U^o, where 1 o and 1 o can be computed by the recursions

$$\text{¸}1\ 1U^o = \ {}^{0}1U^o \ \text{¸} \quad 1U^o \quad 1U^o\text{-} \qquad \text{¸}1\ 1U^o = \quad 1U^o \quad 1U^o \ \text{¸} \ {}^{0}1U^o\text{-}$$

where the recursion is initialized by $_1{}^1U^0 = {}^1U^0$ and $_1{}^1U^0 = {}^1U^0$• Proposition 3.5 thus yields the following counterpart of $\bar{c}{}^1U^0$, but now with an Erlang initial storage level:

$$\sum_{=0}^{l} \frac{{}^1 U^0}{!} \bar{c}^{1\ 0 1}U^0 = \sum_{=0}^{l} \frac{{}^1 U^0}{!} {}^1 \quad {}^1U^0\, \bar{c}{}^1U^0 \ + \quad {}^1U^{00}\,•$$

As argued above, inserting $U = :\,•\,I$ for $:$ large leads to an approximation of the transform conditional on the initial surplus level $-_D{}^10^0$ being equal to the deterministic quantity I• ^

B *Upward jumps.* We subsequently point out how we can deal with upward jumps that are distributed as (i) a mixture of exponentials, and (ii) as an Erlang random variable; the case of a mixture of Erlangs (i.e., distributions from the class) follows upon combining these two procedures.

When the upward jumps are distributed as a mixture of exponentials, the corresponding density can be written as $\sum_{8=1}^{:} 6_8\, 4^{\,`_8 E}$ for some $: 2$ N, positive constants $6_1$-•••-$6_:$, positive parameters $`_1$-•••-$`_:$ (such that $6_1$•$`_1$ + + $6_:$•$`_:$ equals 1), and $E > 0$. It is straightforward to adapt the analysis presented in Section 4.3 to this case. As is readily checked, we have to replace the right-hand side of (4.3) by

$$\sum_{8=1} -\,\frac{6_8}{`_8 \ U}\bar{c}{}^1U^0 \qquad \sum_{8=1} -\,\frac{6_8}{`_8 \ U}\frac{U}{`_8}\bar{c}{}^1{`_8}^0•$$

Mimicking the analysis conducted in Section 4.3, it turns out that one has to slightly adapt the functions ${}^1\,{}^0$ and ${}^1\,{}^0$. Concretely, (i) these functions now have poles at $`_1$-•••-$`_:$, and (ii) the function ${}^1\,{}^0$ contains the unknowns $\bar{c}{}^1{`_1}^0$-•••-$\bar{c}{}^1{`_:}^0$. The resulting $: + 1$ unknowns (i.e., the constants $\bar{c}{}^1{`_1}^0$-•••-$\bar{c}{}^1{`_:}^0$ and $?{}^10^0$) can be determined in the precise same manner as was done in the case of exponentially distributed upward jumps, Indeed, by equating ${}^10^0$ as well as ${}^1{`_8}^0$ to 0 (for all $8 = 1$-•••-$:^0$, we have found $: + 1$ (linear) constraints, from which the $: + 1$ unknowns can be solved in the standard manner.

We conclude this section by discussing the case of the upward jumps being Erlang distributed, say with shape parameter $:$ and scale parameter $`$. It turns out that we have to replace the left-hand side of (4.3) by

$$-\,\int_0^1\int_0^1 \frac{`\,{}^{:}E^{:\,1}}{{}^1:\ \ 1^0!}\,4^{\,`E}\,?{}^1D + E^0\,\mathrm{d}E\ \ U4^{\,UD}\,\mathrm{d}D• \tag{4.11}$$

Applying [4, Lemma A1], we can evaluate (4.11) in terms of $\bar{c}{}^1U^0$ and its derivatives; it simplifies to

$$-\,\left(\frac{`}{`\ \ U}\right)^{:}\bar{c}{}^1U^0 \quad -\,\sum_{=0}^{l}\frac{{}^1\ `^0}{!}\bar{c}^{1\ 0}{}^1{`}^0\left(\left(\frac{`}{`\ \ U}\right)^{:}\ \ 1\right)!\ •$$

Observe that this expression involves $\bar{c}^{10^o 1`o} = \bar{c}^{1`o}$ up to $\bar{c}^{1: \ 1^o 1`o}$, and that at $U = `$ there are poles of multiplicity up to $:$. As before, we eventually find the differential equation $\bar{c}^{0}1U^o = \ 1U^o \bar{c}1U^o \ ¸ \ 1U^o$, but with adapted functions $1U^o$ and $1U^o$. These are now defined by

$$1U^o = \ ^{-}1U^o := \frac{A}{A} \quad \frac{\_}{A}\frac{1 \quad 1 \ 1U^o}{U} \quad \frac{W_1 \ ¸ \ V}{A \ U} \ ¸ \ \frac{\_¸}{A \ U} \quad \frac{` \quad :}{` \quad U} \quad 1 \ -$$

and, using the notation $1 := \bar{c}^{1 \ o 1`o}$,

$$1U^o := \frac{\_}{A}\frac{1 \ 1 \ W_3^o \quad 1 \ 1U \ ¸ \ W_2^o}{U \ ¸ \ W_2 \ ¸ \ W_3} \quad \frac{A}{A} ?1 0^o \ ¸$$
$$\frac{\_¸}{A \ U} \sum_{=0}^{:\tilde{O}1} \frac{1 \quad `o}{!} \quad 1 \quad \frac{` \quad :}{` \quad U} \quad 1 \ •$$

As in Section 4.3, again denoting by $¢1U^o$ the primitive of $1U^o$, the differential equation $\bar{c}^{0}1U^o = \ 1U^o \bar{c}1U^o \ ¸ \ 1U^o$ is solved by

$$\bar{c}1U^o = \int_U^1 1 \ 1[^o \exp \quad ¢1[^o \ d[\ \exp \quad ¢1U^o$$

(if $A \ ¡ \ 0$). In [4, Appendix B] it is explained in detail how the unknowns $?1 0^o$ and $b = 1 1_0 - • • • - 1_{: \ 1}^{o>}$ can be identified. For $: = 2 - 3 - • • •$ this is a rather delicate procedure, due to issues related to the non-analyticity of certain functions.

4.5 Discussion and bibliographical notes

This chapter is based on our paper [4]. In addition to the material we presented in this chapter, that paper also includes the analysis of two related queueing models [4, Sections 3–4], where one of them is the dual of our ruin model, as well as a detailed account of the related literature [4, Section 4.6].

As discussed in the bibliographical notes of Chapter 1, there is an extensive literature on the joint distribution of the ruin time and the corresponding overshoot and undershoot. Key contributions in this area are due to Gerber and Shiu [7, 8], which are put into a broad Lévy-based context in [12]. We also refer to the account in [3, Chapter XII] and to [1], the latter reference also allowing positive jumps.

Although several papers have appeared on the Cramér-Lundberg model with interest, as well as on the Cramér-Lundberg model with additional positive jumps, the combination of both elements is understood considerably less well. For an overview of the main results for ruin models with interest (but without upward jumps, i.e., $\_¸ = 0$), we refer to [3, Section VIII.2], including an explicit result for $?1D - 1 - \mathbf{0}^o$ for the case of exponentially distributed claims. Examples of seminal papers in this area are [5, 10], also covering corresponding diffusion approximations; see also the approximations and bounds in [16]. The characterization of ?¹D° of Lemma 4.1 is in line with [14, Equation (2.1)]; various versions of this result can be found in e.g. [6, 9, 15, 17].

In the literature some attention has been paid to models with jumps in both directions (but without interest, i.e., assuming A = 0). The case of a net cumulative claim process . ¹C° but now with generally distributed downward jumps and phase-type distributed upward jumps (or, alternatively, upward jumps with a distribution that has a rational Laplace transform) has been successfully studied; see e.g. the analysis presented in [13]. This is a useful extension of the traditional Cramér-Lundberg model, recalling that any positive random variable can be approximated arbitrarily closely by a phase-type distributed random variable [2, Thm. III.4.2]. In view of the Wiener-Hopf decomposition (Proposition 1.2), which actually holds for general Lévy processes [11, Chapter VI], this also provides us with results on the case of . ¹C° having phase-type downward jumps and generally distributed upward jumps. Indeed, by flipping the sign of the net cumulative claim process . ¹C° the results of [13] now provide us with the transform of .¹)V°, and as . ¹)V° can be evaluated in a straightforward way, the Wiener-Hopf decomposition yields the transform of .¯¹)V°.

Exercises

4.1 Argue that ¹U° in Remark 4.1 has one positive root. Point out how ?¹0° can be expressed in terms of this root, say U¢. Compute c¹U°, again in terms of U¢.

4.2 ¹¢° Let the negative jumps be exponentially distributed with parameter a ¡ 0, so that 1 ¹U° = a•¹a ¸ U°•
(i) Evaluate ¹U° and ¹U°•
(ii) In the remainder of this exercise we consider the special instance _ = _¸ = A = A = 1• We in addition take $\gamma = 0$ and V = 0, i.e., c̄¹U° corresponds to the transform of the infinite-time ruin probability in the sense that

$$\bar{c}¹U° = U \int_0^1 4^{UD}\, ?¹D°\, \mathrm{d}D•$$

Compute ¢¹U° to show that, with ¹U° := 4^{U¹U ¸ a°} ¹ j` Uj ¹,

$$\bar{c}¹U° = ¹U° \int_U^1 \frac{1}{¹[°} \left(\frac{1}{[¸ a} ?¹0° \frac{1}{`[} \bar{c}¹`° \right) \mathrm{d}[•$$

(iii) Define

$$_0¹G\text{-}H° := ` ¸ a` G ¸ aH\text{-} \quad _1¹G\text{-}H° := 1 ¸ ¹` a° G ¸ H\text{-} \quad _2¹G\text{-}H° := G\text{-}$$

and, for = = 0- 1-•••,

$$, _=¹Uº = \int_U^1 [^{=}4^{\ [}\,\mathrm{d}[= =!4^{\ U} \tilde{Ö}_{:=0} \frac{U^:}{:!}•$$

Show that for U ¡ `

$$\bar{c}¹Uº = \ ¹Uº \tilde{Ö}_{==0}\ _=¹?¹0º-\bar{c}¹`ºº\ ,_=¹Uº•$$

(iv) Argue that the unknowns $?¹0º$ and $\bar{c}¹`º$ satisfy the (linear) system of equations given by

$$\tilde{Ö}_{==0}\ _=¹?¹0º-\bar{c}¹`ºº\ ,_=¹0º = \tilde{Ö}_{==0}\ _=¹?¹0º-\bar{c}¹`ºº=! = 0$$

and

$$\tilde{Ö}_{==0}\ _=¹?¹0º-\bar{c}¹`ºº\ ,_=¹`º = 0•$$

(v) Verify that

$$?¹0º = \frac{` ¸ a}{a^2 ¸ `a ¸ 2a ¸ `}- \quad \bar{c}¹`º = \frac{`}{a^2 ¸ `a ¸ 2a ¸ `}•$$

4.3 In Remark 4.3 it is stated that when taking the limit U ! ` one indeed obtains $\bar{c}¹`º$. Verify this.

4.4 Take $W_1 = W_2 = W_3 = 0$. In addition, let $\_{¸} = 0$ and $A = 0$. This is a setting that directly relates to the model that will be discussed in Chapter 6.
(i) First consider $V = 0$. Use the differential equation (4.4) to show that

$$\bar{c}¹Uº = 1 \quad \exp \quad \frac{\_}{A}\int_0^U \frac{1 \quad 1\ ¹Gº}{G}\mathrm{d}G\ •$$

(ii) Now let V ¡ 0. Use Remark 4.4 to show that

$$\bar{c}¹Uº = \int_0^U \frac{[}{U}^{V•A} \frac{\_}{A}\frac{1 \quad 1\ ¹[º}{[} \exp \int_{[}^U \frac{\_}{A}\frac{1 \quad 1\ ¹Gº}{G}\mathrm{d}G\ \mathrm{d}[•$$

(iii) Simplify the latter integral further in the case that $1\ ¹Uº = a•¹a ¸ Uº$, i.e., the case of exponentially distributed negative jumps.

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Chapter 5
Threshold-based net cumulative claim process

Abstract – This chapter discusses a variant of the Cramér-Lundberg model in which the net cumulative claim process obeys different stochastic dynamics (in terms of the claim arrival rate, premium rate, and claim-size distribution) above and below a threshold x. For this setting of a threshold-based net cumulative claim process we evaluate the ruin probability over an exponentially distributed interval. An important role is played by the concept of scale functions.

5.1 Introduction

In this chapter we consider the setting in which the net cumulative claim process displays different behavior above and below the threshold $x \in (-\infty, u)$, with $u > 0$ denoting the initial reserve level. We denote the resulting net cumulative claim process by $Y_x(t)$ and its running maximum process by $\bar{Y}_x(t)$, with the objective to evaluate the time-dependent ruin probability, i.e., $p(u, x, t) := \mathbb{P}(\bar{Y}_x(t) > u)$.

In more concrete terms, the model with such a threshold-based net cumulative claim process can be described as follows. Below x the claim arrival rate is λ_-, the premium rate is r_- and the claims have Laplace-Stieltjes transform $b_-(\alpha)$ (also when the claim under consideration is such that due to the corresponding jump the process $\bar{Y}_x(t)$ exceeds x), whereas above x the claim arrival rate is λ_+, the premium rate is r_+ and the claims have Laplace-Stieltjes transform $b_+(\alpha)$. We focus on the (somewhat more complicated) variant that $x \in (0, u)$, where it is noted that the case that $x \in (-\infty, 0]$ can be dealt with analogously. It is remarked that the meaning of the rates λ_- and λ_+ differs from the one that has been used in Chapter 4.

In our analysis, we (as before) focus on the probability $p(u, x, T_\beta)$ of ruin before an exponentially distributed epoch T_β (sampled independently of the threshold-based net cumulative claim process). An important role is played by the concept of *scale functions*, discussed in Section 5.2. Then, in Section 5.3, these scale functions are used to evaluate a decomposition of $p(u, x, T_\beta)$. Finally, in Section 5.4 we determine various additional auxiliary quantities.

5.2 Scale functions

Consider the net cumulative claim process $Y(t)$ in the non-threshold setting, i.e., with claim arrival rate λ, premium rate r, and the claim-size distribution having Laplace-Stieltjes transform $b(\alpha)$. In this section the focus is on computing the objects, for $x \geqslant 0$, $x_+ > 0$ and $q > 0$,

$$\pi(x, x_+, q) := \mathbb{P}(\sigma(x) < \min\{\tau(x_+), T_q\}),$$
$$\pi_+(x, x_+, q) := \mathbb{P}(\tau(x_+) < \min\{\sigma(x), T_q\});$$

here $\tau(x_+)$ is defined (as before) as the first epoch that $Y(t)$ enters (x_+, ∞), whereas $\sigma(x)$ is the first epoch that $Y(t)$ enters $(-\infty, -x]$ (where it is observed that $Y(\sigma(x)) = -x$, as in the downward direction every level is crossed with equality). The Laplace exponent $\varphi(\alpha)$ of the process $Y(t)$ is defined in the usual manner: $\varphi(\alpha) = r\alpha - \lambda(1 - b(\alpha))$. In the next sections we use the identified expressions for $\pi(x, x_+, q)$ and $\pi_+(x, x_+, q)$ to evaluate the probability $p(x, y, T_q)$ in the model with a threshold-based net cumulative claim process.

▷ *Definition of the scale function, and some properties.* Now define the scale function $W^{(q)}(x)$, with $x \geqslant 0$, as the function whose Laplace transform is a given function: for $q < \varphi(\alpha)$,

$$\int_0^\infty e^{-\alpha x} W^{(q)}(x)\,\mathrm{d}x = \frac{1}{\varphi(\alpha) - q}. \tag{5.1}$$

We do not comment on the existence of such a function $W^{(q)}(x)$; an extensive analysis is presented (for the larger class of spectrally one-sided Lévy processes) in e.g. [10, Section 8.2]. There is in addition a second scale function, defined as

$$Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y)\,\mathrm{d}y,$$

so that, swapping the order of the integrals, for $q < \varphi(\alpha)$,

$$\begin{aligned}\int_0^\infty e^{-\alpha x} Z^{(q)}(x)\,\mathrm{d}x &= \frac{1}{\alpha} + q \int_0^\infty \int_y^\infty e^{-\alpha x} W^{(q)}(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{\alpha} + \frac{q}{\alpha}\,\frac{1}{\varphi(\alpha) - q}.\end{aligned} \tag{5.2}$$

In Chapter 1 we characterized the distribution of $\bar{Y}(T_q)$ in terms of the Laplace exponent $\varphi(\alpha)$ and its right-inverse $\psi(q)$. The next lemma, that will play a crucial role in determining $\pi_+(x, x_+, q)$, shows that using the concept of scale functions, we can develop an alternative representation.

Lemma 5.1 *For any $x \geqslant 0$ and $q > 0$,*

$$\mathbb{P}(\bar{Y}(T_q) \geqslant x) = Z^{(q)}(x) - \frac{q}{\psi(q)}\, W^{(q)}(x).$$

Proof. Proving this claim amounts to verifying that the transform (with respect to u, that is) coincides with (1.4); here we use the fact that there is a one-to-one correspondence between a function and its Laplace transform, cf. Appendix A.3. The fact that two transforms are equal follows immediately, using Equations (5.1) and (5.2); see Exercise 5.2.

The following lemma presents useful alternative expressions for the target quantities $\xi(u, u_+, \beta)$ and $\xi_+(u, u_+, \beta)$.

Lemma 5.2 *For any* $u \geqslant 0$, $u_+ > 0$, *and* $\beta > 0$,

$$\xi(u, u_+, \beta) = \mathbb{E}\, e^{-\beta\tau(u)}\, 1\{\tau(u) < \sigma(u_+)\},$$
$$\xi_+(u, u_+, \beta) = \mathbb{E}\, e^{-\beta\sigma(u_+)}\, 1\{\sigma(u_+) < \tau(u)\}.$$

Proof. We establish the claim for $\xi(u, u_+, \beta)$; the other claim follows analogously. Applying integration by parts,

$$\begin{aligned}\xi(u, u_+, \beta) &= \int_0^\infty \beta e^{-\beta t}\, \mathbb{P}(\tau(u) < t, \tau(u) < \sigma(u_+))\, \mathrm{d}t \\ &= \int_0^\infty e^{-\beta t}\, \mathbb{P}(\tau(u) \in \mathrm{d}t, \tau(u) < \sigma(u_+)) \\ &= \mathbb{E}\, e^{-\beta\tau(u)}\, 1\{\tau(u) < \sigma(u_+)\}.\end{aligned}$$

Alternatively, one could replace the first line in the above formula by

$$\xi(u, u_+, \beta) = \int_0^\infty \mathbb{P}(T_\beta \geqslant t)\, \mathbb{P}(\tau(u) \in \mathrm{d}t, \tau(u) < \sigma(u_+)),$$

with no need for integration by parts.

▷ *Evaluation of* $\xi(u, u_+, \beta)$. Our next objective is to find an expression for $\xi(u, u_+, \beta)$ in terms of the scale functions. We do so by first finding, in Lemma 5.3, an expression for $\xi(u, u_+, 0)$ in the case that the net cumulative claim process has a negative expectation. This result is then 'bootstrapped' to an expression for $\xi(u, u_+, \beta)$ in Theorem 5.1.

Lemma 5.3 *Assume* $\mathbb{E}\, X(1) < 0$, *or equivalently* $\varphi'(0) \geqslant 0$. *Then, for any* $u \geqslant 0$, $u_+ > 0$,

$$\xi(u, u_+, 0) = \frac{W^{(0)}(u_+)}{W^{(0)}(u_+ + u)}.$$

Proof. The key element in this proof is the identity

$$\mathbb{P}(\bar{X}(\infty) < u_+) = \mathbb{P}(\bar{X}(\infty) < u_+ + u)\, \mathbb{P}(\tau(u) < \sigma(u_+)),$$

where it is used that $X(\tau(u)) = -u$ in combination with the strong Markov property; in addition, it is noted that because of $\mathbb{E}\, X(1) < 0$ both probabilities $\mathbb{P}(\bar{X}(\infty) < u_+)$ and $\mathbb{P}(\bar{X}(\infty) < u_+ + u)$ are strictly positive. We conclude that

$$\pi(u, u_+, 0) = \frac{\mathbb{P}(\bar{Y}(\infty) \le u_+)}{\mathbb{P}(\bar{Y}(\infty) \le u_+ + u)}.$$

Hence it is left to prove that, for $u \ge 0$, $\mathbb{P}(\bar{Y}(\infty) \le u)$ is proportional to $W^{(0)}(u)$. This we show by establishing that the transforms of both objects are proportional. Indeed, exploiting the results of Chapter 1 (and in particular Corollary 1.1),

$$\int_{0+}^{\infty} e^{-\alpha u}\,\mathbb{P}(\bar{Y}(\infty) < u)\,\mathrm{d}u = \frac{1}{\alpha}\mathbb{P}(\bar{Y}(\infty) = 0) + \frac{1}{\alpha}\int_{0+}^{\infty} e^{-\alpha u}\,\mathbb{P}(\bar{Y}(\infty) \in \mathrm{d}u)$$
$$= \frac{1}{\alpha}\int_{0}^{\infty} e^{-\alpha u}\,\mathbb{P}(\bar{Y}(\infty) \in \mathrm{d}u) = \frac{\varphi'(0)}{\varphi(\alpha)},$$

which is proportional to $1/\varphi(\alpha)$, i.e., the transform of $W^{(0)}(u)$.

Theorem 5.1 *For any $u \ge 0$, $u_+ > 0$ and $\beta \ge 0$,*

$$\pi(u, u_+, \beta) = \frac{W^{(\beta)}(u_+)}{W^{(\beta)}(u_+ + u)}.$$

Proof. The main idea is to study the process $Y(t)$, for $\beta \ge 0$, under an exponential change-of-measure with 'twist' $-\psi(\beta) < 0$. More concretely, calling this alternative probability model $\mathbb{Q}$, the Laplace exponent under $\mathbb{Q}$ is

$$\varphi_{\mathbb{Q}}(\alpha) = \varphi(\alpha + \psi(\beta)) - \varphi(\psi(\beta)) = \varphi(\alpha + \psi(\beta)) - \beta. \tag{5.3}$$

It is clear that we have constructed a process with negative mean: the fact that $\varphi'(\psi(0)) \ge 0$, in combination with the facts that (i) the right inverse $\psi(\beta)$ is increasing in β and (ii) $\varphi(\alpha)$ is increasing for $\alpha \ge \psi(\beta)$, yields, in self-evident notation,

$$\mathbb{E}_{\mathbb{Q}} Y(1) = -\varphi'_{\mathbb{Q}}(0) = -\varphi'(\psi(\beta)) < 0;$$

see Figure 5.1 for a pictorial illustration.

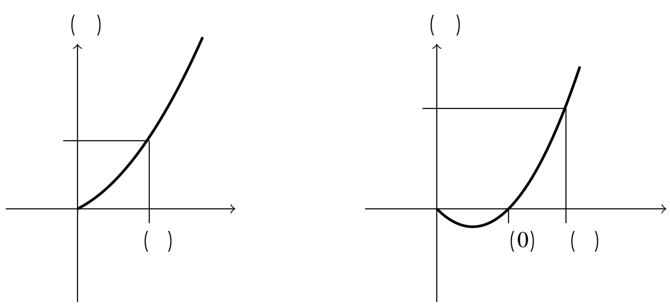

Fig. 5.1 The functions $\varphi(\alpha)$ with $\varphi'(0) \ge 0$ (left panel) and $\varphi(\alpha)$ with $\varphi'(0) < 0$ (right panel). In the former case $\psi(\beta) \ge \psi(0) = 0$, whereas in the latter case $\psi(\beta) \ge \psi(0) \ge 0$. Observe that in both cases $\varphi'(\psi(\beta)) \ge 0$.

As is readily verified, the likelihood ratio connecting $\mathbb{P}$ and $\mathbb{Q}$ is such that,

$$\frac{\mathbb{P}(X(t)\in \mathrm{d}x)}{\mathbb{Q}(X(t)\in \mathrm{d}x)} = e^{\beta t}e^{\psi(\beta)x}; \tag{5.4}$$

to see this, observe that due to (5.3),

$$\int_{-\infty}^{\infty} e^{-\alpha x}\,\mathbb{Q}(X(t)\in \mathrm{d}x) = \mathbb{E}_{\mathbb{Q}}\, e^{-\alpha X(t)} = \mathbb{E}\, e^{-(\alpha+\psi(\beta))X(t)-\beta t}$$
$$= e^{-\beta t}\int_{-\infty}^{\infty} e^{-(\alpha+\psi(\beta))x}\,\mathbb{P}(X(t)\in \mathrm{d}x).$$

We conclude from Lemma 5.3, which we can apply because $\mathbb{E}_{\mathbb{Q}}X(1) < 0$, that

$$\mathbb{Q}(\sigma(u) < \tau(u_+)) = \frac{\mathbb{Q}(\bar{X}(\infty) < u_+)}{\mathbb{Q}(\bar{X}(\infty) < u_+ + u)}.$$

On the other hand, by applying the expression for the likelihood ratio that was given in (5.4) in combination with Lemma 5.2, we obtain that

$$\mathbb{Q}(\sigma(u) < \tau(u_+)) = \mathbb{E}\, e^{-\beta\sigma(u)}e^{-\psi(\beta)X(\sigma(u))}1\{\sigma(u) < \tau(u_+)\}$$
$$= e^{\psi(\beta)u}\,\delta(u, u_+, \beta).$$

Upon combining the above findings, we derive the following expression for the quantity of our interest:

$$\delta(u, u_+, \beta) = e^{-\psi(\beta)u}\,\frac{\mathbb{Q}(\bar{X}(\infty) < u_+)}{\mathbb{Q}(\bar{X}(\infty) < u_+ + u)}$$
$$= \frac{e^{\psi(\beta)u_+}\,\mathbb{Q}(\bar{X}(\infty) < u_+)}{e^{\psi(\beta)(u_+ + u)}\,\mathbb{Q}(\bar{X}(\infty) < u_+ + u)}.$$

We thus conclude that we have proven the stated once we have established that $e^{\psi(\beta)u}\,\mathbb{Q}(\bar{X}(\infty) < u)$ is proportional to $W^{(\beta)}(u)$. This we show by proving that their respective transforms match up to a multiplicative constant. A computation similar to the one featuring in Lemma 5.3 reveals that

$$\int_{0+}^{\infty} e^{-\alpha u}e^{\psi(\beta)u}\,\mathbb{Q}(\bar{X}(\infty) < u)\,\mathrm{d}u = \frac{\varphi_{\mathbb{Q}}'(0)}{\varphi_{\mathbb{Q}}(\alpha - \psi(\beta))} = \frac{\varphi'(\psi(\beta))}{\varphi(\alpha) - \beta},$$

which is proportional to $1/(\varphi(\alpha) - \beta)$. The proof is concluded by recalling from Equation (5.1) that $1/(\varphi(\alpha) - \beta)$ is the transform of $W^{(\beta)}(u)$.

▷ *Evaluation of* $\delta_+(u, u_+, \beta)$. We now focus on the evaluation of the second quantity, i.e., $\delta_+(u, u_+, \beta)$. As it turns out, here Lemma 5.1 plays a crucial role.

Theorem 5.2 *For any* $u \geq 0$, $u_+ > 0$ *and* $\beta \geq 0$,

$$\xi_+(u, u_+, \beta) = Z^{(\beta)}(u_+) - Z^{(\beta)}(u_+ - u)\,\frac{W^{(\beta)}(u_+)}{W^{(\beta)}(u_+ - u)}. \tag{5.5}$$

Proof. We first decompose

$$\xi_+(u, u_+, \beta) = \mathbb{E}\, e^{-\beta\tau(u_+)} 1\{\tau(u_+) < \infty\} - \mathbb{E}\, e^{-\beta\tau(u_+)} 1\{\sigma(u) \leq \tau(u_+)\}.$$

The first term in the right-hand side reads, using (1.2) and the equivalence of the events $\{\tau(u) \leq t\}$ and $\{\bar{Y}(t) > u\}$,

$$\mathbb{E}\, e^{-\beta\tau(u_+)} 1\{\tau(u_+) < \infty\} = \mathbb{P}(\tau(u_+) \leq T_\beta) = \mathbb{P}(\bar{Y}(T_\beta) > u_+),$$

which we can evaluate in terms of scale functions relying on Lemma 5.1. In addition, using the strong Markov property,

$$\begin{aligned}
&\mathbb{E}\, e^{-\beta\tau(u_+)} 1\{\sigma(u) \leq \tau(u_+)\}\\
&\quad= \mathbb{P}(\sigma(u) \leq \tau(u_+) \leq T_\beta)\\
&\quad= \mathbb{P}(\sigma(u) \leq T_\beta, \sigma(u) \leq \tau(u_+))\,\mathbb{P}(\tau(u_+ - u) \leq T_\beta).
\end{aligned}$$

Upon combining the above findings, we obtain

$$\xi_+(u, u_+, \beta) = \mathbb{P}(\bar{Y}(T_\beta) > u_+) - \xi(u, u_+, \beta)\,\mathbb{P}(\bar{Y}(T_\beta) > u_+ - u).$$

It is directly verified, using Theorem 5.1 and Lemma 5.1, that this yields the desired expression. Here we use that

$$\begin{aligned}
&\mathbb{P}(\bar{Y}(T_\beta) > u_+) - \xi(u, u_+, \beta)\,\mathbb{P}(\bar{Y}(T_\beta) > u_+ - u)\\
&\quad= Z^{(\beta)}(u_+) - \frac{\beta}{\Phi(\beta)}\, W^{(\beta)}(u_+)\\
&\qquad- \frac{W^{(\beta)}(u_+)}{W^{(\beta)}(u_+ - u)}\left(Z^{(\beta)}(u_+ - u) - \frac{\beta}{\Phi(\beta)}\, W^{(\beta)}(u_+ - u)\right),
\end{aligned}$$

which equals the right-hand side of (5.5).

In Exercise 5.5 we demonstrate the power of scale functions, by evaluating the transform of the workload distribution in the finite-capacity M/G/1 queue.

5.3 Decomposition

As pointed out earlier, our interest lies in evaluating $p(u, v, T_\beta)$, the probability of $Y_v(t)$ exceeding level u before time T_β (which is, as before, an exponentially distributed random variable with parameter β, independent of everything else), with $Y_v(0) = 0$ and v denoting the threshold. This section presents a decomposition by which $p(u, v, T_\beta)$ can be calculated.

B *Introduction of required objects.* In our analysis a key quantity is the first passage time

$$g(F) := \inf\{C > 0 : ._E(C) \geq F \mid ._E(0) = 0\}.$$

In addition, for $H \in (E, D)$,

$$g_H(D) := \inf\{C > 0 : ._E(C) \geq D \mid ._E(0) = H\},$$
$$f_H(E) := \inf\{C > 0 : ._E(C) \leq E \mid ._E(0) = H\}$$

(where it is observed that in the definition of the ruin probability $?(D, E, C)$ it was evidently assumed that $._E(0) = 0$). It is noted that in the definition of $f_H(E)$ we could have replaced '$\leq E$' by '$= E$'.

In the analysis a crucial role is played by the overshoot over level E, jointly with the indicator function of the event of $._E(C)$ actually exceeding E before time $)_V$. To this end, we introduce the object

$$:(E, C, W) := \mathbb{E}\, 4^{W(._E(g(E)) - E)} 1\{g(E) \leq C\},$$

where $._E(g(E)) - E$ is to be interpreted as the overshoot over level E. In Section 5.4, we point out how to evaluate the double transform of $:(E, C, W)$, or, equivalently,

$$\hat{}(U, V, W) := \int_0^{\infty} 4^{-UE} :(E,)_V, W)\, \mathrm{d}E.$$

This, applying Laplace inversion, allows us to evaluate

$$\mathbb{P}(._E(g(E)) - E \in \mathrm{d}H, g(E) \leq)_V);$$

in the sequel we call the corresponding density (E, H, V).

We proceed by introducing two probabilities, $@(D, E, C)$ and $\bar{@}(D, E, C)$, with $E < D$. Here $@(D, E, C)$ is to be interpreted as the probability that starting in E, first a level above E is attained (before C), and then E is reached again (before C), before D is exceeded (also before C). Formally, $@(D, E, C) := \mathbb{P}((D, E, C) \mid ._E(0) = E)$, with

$$(D, E, C) := \left\{ \begin{array}{l} B_1 := \inf\{B \geq 0 : ._E(B) \geq E\} \leq C, \\ B_2 := \inf\{B \geq B_1 : ._E(B) = E\} \leq C, \\ \exists B \in (B_1, B_2) : ._E(B) \leq D \end{array} \right\}.$$

Likewise, $\bar{@}(D, E, C)$ represents the probability that starting in E, first a level above E is attained (before C), and then level D is exceeded (also before C), before E is reached from above (also before C). Formally, $\bar{@}(D, E, C) := \mathbb{P}((D, E, C) \mid ._E(0) = 0)$, with

$$(D, E, C) := \left\{ \begin{array}{l} B_1 := \inf\{B \geq 0 : ._E(B) \geq E\} \leq C, \\ B_2 := \inf\{B > B_1 : ._E(B) > D\} \leq C, \\ \exists B \in [B_1, B_2] : ._E(B) \geq E \end{array} \right\}.$$

Notice that this scenario also includes the case in which at the first time level E is exceeded, level D is exceeded as well. The probabilities $@(D,E,)_V)$ and $@(D,E,)_V)$ are evaluated in Section 5.4.

B *Evaluation of the ruin probability.* We now evaluate our target probability $?(D,E,)_V)$. To this end, realize that there are essentially three (disjoint) ways to exceed D, starting with $._E(0) = 0$; we refer to Figure 5.2 for a pictorial illustration.

1. In the first place, the level E can be exceeded (before $)_V$) with an overshoot that is larger than $D \quad E$. This leads to the contribution

$$?_1(D,E,)_V) := \int_{D\ E}^{1} (E,H,V)\,\mathrm{d}H.$$

2. In the second place, there is the scenario that the level E is exceeded with an overshoot that lies between 0 and $D \quad E$, but that from that point on D is exceeded before E is reached (and all these events before $)_V$). This corresponds to the contribution

$$?_2(D,E,)_V) := \int_0^{D\ E} (E,H,V)\,X_{¸,H}(D,E,V)\,\mathrm{d}H,$$

   with the probability $X_{¸,H}(D,E,V) := \mathrm{P}(g_H(D) \leq \min\{f_H(E),)_V\})$ to be evaluated in Section 5.4.

3. In the third place, the level E can be exceeded with an overshoot that lies between 0 and $D \quad E$, but from that point on E is reached before D is exceeded (and all these events occur before $)_V$). From that point on there is a geometric number of attempts of exceeding D starting at level E; in each of these attempts, the process first has to exceed level E again, and after that D should be exceeded before returning to E (all these events occurring before $)_V$); cf. the black dots in the bottom panel of Figure 5.2. This leads to the contribution

$$\begin{aligned}
&?_3(D,E,)_V)\\
&:= \int_0^{D\ E} (E,H,V)\,X_{\ ,H}(D,E,V)\,\mathrm{d}H \sum_{:=0}^{1} @(D,E,)_V)\ @(D,E,)_V)^{:}\\
&= \frac{@(D,E,)_V)}{1 \quad @(D,E,)_V)} \int_0^{D\ E} (E,H,V)\,X_{\ ,H}(D,E,V)\,\mathrm{d}H,
\end{aligned}$$

   with the probability $X_{\ ,H}(D,E,V) := \mathrm{P}(f_H(E) \leq \min\{g_H(D),)_V\})$ being evaluated in Section 5.4.

Combining the above, we have found the following decomposition result.

Theorem 5.3 *For any $D > 0$, $E \in (0,D)$, and $V > 0$,*

$$?(D,E,)_V) = ?_1(D,E,)_V) + ?_2(D,E,)_V) + ?_3(D,E,)_V).$$

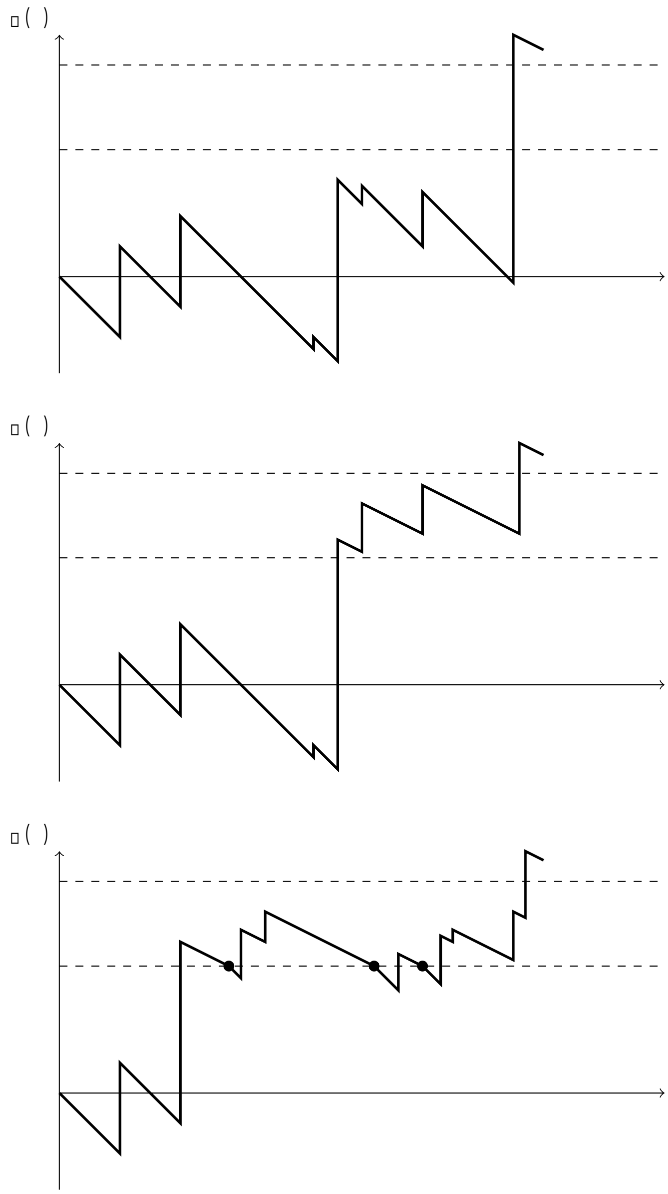

Fig. 5.2 Net cumulative claim process $._E{}^1C^{\circ}$. The top panel corresponds to Scenario 1, the middle panel to Scenario 2, and the bottom panel to Scenario 3 (with the black dots indicating the start of a new attempt to exceed level D, starting at level E, each of them succeeding with probability $@^1D-E-)_V{}^{\circ}$).

5.4 Computation of auxiliary objects

In this section we point out how to evaluate the objects featuring in the decomposition presented in Theorem 5.3: the density $f(v,x,q)$ (through the associated transform $\kappa(\alpha,q,\vartheta)$), the probabilities $p_{-,v}(u,x,q)$ and $p_{+,v}(u,x,q)$, and the probabilities $\pi(u,x,\tau_q)$ and $\pi(u,x,\tau_q)$.

▷ *Evaluation of* $\kappa(\alpha,q,\vartheta)$. We follow a procedure that is similar to the one used in Exercise 1.2. For conciseness, we write $\xi(x)$ to denote $\xi(x,\tau_q,\vartheta)$. Indeed, as $\Delta t\downarrow 0$, with B denoting a generic claim level when the net cumulative claim process is below x,

$$\xi(x)=\lambda\,\Delta t\int_0^x \mathbb{P}(B\in dy)\,\xi(x-y)+\lambda\,\Delta t\int_x^\infty \mathbb{P}(B\in dy)\,e^{-\vartheta(y-x)}+(1-\lambda\,\Delta t-q\,\Delta t)\,\xi(x+r\,\Delta t)+o(\Delta t).$$

Subtracting $\xi(x+r\,\Delta t)$ from both sides, dividing by Δt, and sending Δt to 0, we obtain the integro-differential equation

$$\xi'(x)\,r=-\lambda\int_0^x \mathbb{P}(B\in dy)\,\xi(x-y)-\lambda\int_x^\infty \mathbb{P}(B\in dy)\,e^{-\vartheta(y-x)}+(\lambda+q)\,\xi(x).$$

The next step is to write this equation in terms of the transform $\kappa(\alpha,q,\vartheta)$. To this end, we multiply the entire equation by $e^{-\alpha x}$ and integrate over x, so as to obtain

$$r\,\xi(0)-r\,\alpha\,\kappa(\alpha)=\lambda\,\kappa(\alpha)\,b(\alpha)+\lambda\,\frac{b(\alpha)-b(\vartheta)}{\vartheta-\alpha}-(\lambda+q)\,\kappa(\alpha),$$

where we locally write $\kappa(\alpha)$ to denote $\kappa(\alpha,q,\vartheta)$. Solving $\kappa(\alpha)$ from the above equation, and writing $\varphi(\alpha):=r\alpha-\lambda(1-b(\alpha))$,

$$\kappa(\alpha)=\frac{1}{\varphi(\alpha)-q}\left(r\,\xi(0)-\lambda\,\frac{b(\alpha)-b(\vartheta)}{\vartheta-\alpha}\right).$$

With $\psi(q)$ the right inverse of $\varphi(\alpha)$, we observe that the denominator is zero for $\alpha=\psi(q)$, and hence the numerator as well. We conclude that

$$\kappa(\alpha,q,\vartheta)=\frac{\lambda}{\varphi(\alpha)-q}\left(\frac{b(\psi(q))-b(\vartheta)}{\vartheta-\psi(q)}-\frac{b(\alpha)-b(\vartheta)}{\vartheta-\alpha}\right).$$

▷ *Evaluation of* $p_{-,v}(u,x,q)$ *and* $p_{+,v}(u,x,q)$. Here we use the theory developed in Section 5.2. We obtain from Theorem 5.1 that

$$X_{-H}(D,E,V) = \frac{,_{¸}^{(V)}(D \quad H)}{,_{¸}^{(V)}(D \quad E)},$$

with $,_{¸}^{(V)}(D)$ such that

$$\int_0^{\infty} 4^{UD}\, ,_{¸}^{(V)}(D)\,\mathrm{d}D = \frac{1}{i_{¸}(U) \quad V},$$

where $i_{¸}(U) := A_{¸}U \quad \_{¸}(1 \quad 1_{¸}(U))$. Also, with $/_{¸}^{(V)}(D)$ defined in the evident manner, from Theorem 5.2,

$$X_{¸,H}(D,E,V) = /_{¸}^{(V)}(D \quad H) \quad /_{¸}^{(V)}(D \quad E)\frac{,_{¸}^{(V)}(D \quad H)}{,_{¸}^{(V)}(D \quad E)}.$$

B *Evaluation of* $@(D,E,)_V)$ *and* $@(D,E,)_V)$. With $X_{-H}(D,E,V)$ and $X_{¸,H}(D,E,V)$ as given above, it is easily verified that

$$@(D,E,)_V) = \int_0^{D\ E} (0_{¸},H,)_V)\, X_{¸,H}(D,E,V)\,\mathrm{d}H \ ¸ \int_{D\ E}^{\infty} (0_{¸},H,)_V)\,\mathrm{d}H$$

and

$$@(D,E,)_V) = \int_0^{D\ E} (0_{¸},H,)_V)\, X_{-H}(D,E,V)\,\mathrm{d}H.$$

The density $(0_{¸},H,)_V)$ can be determined as pointed out above.

5.5 Discussion and bibliographical notes

The most basic model with a threshold-based net cumulative claim process is the one in which only the premium rate A depends on being below or above the level E, the so-called *refracted process*, the resulting process often being referred to as the *threshold dividend model.* An analysis of this model, focusing on the all-time ruin probability, can be found in [2, Section VIII.1a]; see also for instance [9, 12, 14]. An extension to a general Lévy setup is presented in [11]. The extension to multiple thresholds is treated in detail in [1] and [2, Section VIII.1b]; cf. also [13].

A model that slightly differs from the one of the present chapter is analyzed in [5, 6]. In the model of [5] the decision to adapt the premium rate only takes place at claim arrival instants, while in [6] the premium rate can be adapted at Poisson observer instants. The ruin probability is expressed through a system of integro-differential equations, which can be solved in the case of exponentially distributed claim sizes.

In the queueing literature various related models have been studied, most notably M/G/1-type queues with different dynamics below and above a threshold. References that are relevant in this context include [3, 4]. The M/G/1 queue with service speed adaptations is covered by e.g. [7, 8].

In this chapter the concept of scale function plays a key role. More background on these scale functions in the context of the class of spectrally one-sided Lévy processes, and their role in Lévy fluctuation theory, is given in [10, Chapter VIII].

Exercises

5.1 In this exercise we focus on the scale function in the case that the claim sizes are exponentially distributed with parameter μ. More precisely, we consider the scale function $W^{(q)}(x)$, implicitly defined by (5.1).
(i) Show that

$$\bar W(\alpha) := \int_0^\infty e^{-\alpha x}\, W^{(q)}(x)\,\mathrm{d}x = \frac{\mu+\alpha}{r\alpha^2 + s\alpha - q\mu},$$

with $s := r\mu - \lambda - q$.
(ii) With

$$\alpha_\pm := \frac{1}{2r}\left(-s \pm \sqrt{s^2 + 4rq\mu}\right),$$

prove that

$$\bar W(\alpha) = \frac{1}{r}\frac{\mu+\alpha_+}{\alpha_+-\alpha_-}\frac{1}{\alpha-\alpha_+} + \frac{1}{r}\frac{\mu+\alpha_-}{\alpha_--\alpha_+}\frac{1}{\alpha-\alpha_-}.$$

(iii) Determine $W^{(q)}(x)$.
(iv) Use this to determine $X(x, x_+, q)$ and $X_+(x, x_+, q)$.

5.2 Give a detailed proof of Lemma 5.1.

5.3 Consider the case that below the threshold u the claim sizes have an exponential distribution, i.e., $\bar b(\alpha) = \mu/(\mu+\alpha)$.
(i) Argue that $\pi(u, y, q) = \mu e^{-\mu y}\,\mathbb{P}(g(u) \leq T_q)$, and that

$$\hat\pi(\alpha, q, \beta) = \frac{\mu}{\mu+\beta}\int_0^\infty e^{-\alpha u}\,\mathbb{P}(g(u) \leq T_q)\,\mathrm{d}u.$$

(ii) Use the duality between the events $\{g(u) \leq T_q\}$ and $\{\bar X(T_q) \geq u\}$ to prove that

$$\hat\pi(\alpha, q, \beta) = \frac{\mu}{\mu+\beta}\,\frac{1}{\phi(\alpha) - q}\,\frac{\phi(\alpha)}{\alpha}\,\frac{q}{\psi(q)},$$

with

$$\phi(\alpha) = \alpha r - \frac{\lambda\alpha}{\mu+\alpha},$$

and $\psi(q)$ its right-inverse.

5.4 This exercise is about the M/G/1 workload process reaching the value $u > 0$ before the exponentially distributed time T_β. We consider the model discussed in Chapter 1, i.e., we are not in the setting of a threshold-based net cumulative claim process. This concretely means that the workload's driving process $Y(t)$ corresponds to an arrival rate λ, job sizes distributed as the random variable B with LST $b(\alpha)$ and service rate r. As in Exercise 1.7, we assume that the workload process starts at an arbitrary level x.

The main object of interest is

$$\varphi_x(u,\beta) := \mathbb{P}(\exists s \in [0,T_\beta] : Q(s) > u \mid Q(0) = x),$$

for $x \in [0,u)$. The underlying idea of this exercise is to show that this quantity can be expressed using the concept of scale functions. We do this through the probabilities of the type $\xi_-(u_-,u_+,\beta)$ and $\xi_+(u_-,u_+,\beta)$ introduced in Section 5.2 that were given in terms of scale functions in Theorems 5.1 and 5.2.

(i) Prove that

$$\begin{aligned}\varphi_x(u,\beta) = {} & \xi_+(x,u-x,\beta) + \xi_-(x,u-x,\beta)\,\frac{\lambda}{\lambda+\beta}\Bigg(\int_u^\infty \mathbb{P}(B \in \mathrm{d}v) + \int_0^u \mathbb{P}(B \in \mathrm{d}v)\,\xi_+(v,u-v,\beta) \\ & + \int_0^u \mathbb{P}(B \in \mathrm{d}v)\,\xi_-(v,u-v,\beta)\,\varphi_0(u,\beta)\Bigg).\end{aligned}$$

(ii) Observe that the only unknown left is $\varphi_0(u,\beta)$, i.e., the probability of the workload exceeding u before T_β, given that the system started empty. Argue, using that $\xi_-(0,u,\beta) = 1$ and $\xi_+(0,u,\beta) = 0$, that

$$\varphi_0(u,\beta) = \frac{\lambda\int_u^\infty \mathbb{P}(B \in \mathrm{d}v) + \lambda\int_0^u \mathbb{P}(B \in \mathrm{d}v)\,\xi_+(v,u-v,\beta)}{\lambda+\beta-\lambda\int_0^u \mathbb{P}(B \in \mathrm{d}v)\,\xi_-(v,u-v,\beta)}.$$

(iii) Show that $\varphi_0(u,\beta) \to 1$ as $\beta \downarrow 0$, and hence that $\varphi_x(u,\beta) \to 1$ as $\beta \downarrow 0$ for all $x \in [0,u)$. Interpret this finding.

5.5 $(\ast)$ In this exercise we consider the workload in an M/G/1 system in which this workload is truncated at $K > 0$. We focus on *partial rejection*: if due to a certain arriving job the level K would be exceeded, the part that still fits in the buffer is allowed into the system. We focus on, for $x \in [0,K]$,

$$\psi(x,\alpha,\beta) \equiv \psi(x) := \mathbb{E}_x e^{-\alpha Q(T_\beta)} := \mathbb{E}(e^{-\alpha Q(T_\beta)} \mid Q(0) = x).$$

(i) Argue that, for $x \in (0,K]$,

j ¹Gº = X¸¹Gº j¹ º ¸ X ¹Gº j¹0º ¸ X¢¹Gº-

where X¸¹Gº := X¸¹G- G- Vº- X ¹Gº := X ¹G- G- Vº- and

X¢¹Gº := 4 UG E 4 U. ¹)Vº 1f)V Ÿ minf f ¹Gº- g¹ Gºgg •

(ii) Show that

X¢¹Gº = 4 UG V / (V i ¹Uº) ¹minf f ¹Gº- g¹ Gºgº -

where, for C > 0, G ¡ 0 and H > 0,

¹Cº := E 4 U. ¹)Vº 1f)V i Cg •

(*Hint*: cf. the approach followed in Exercise 1.8.)

(iii) Prove that

¹ f ¹Gºº = 4UG 4 k¹Vº G V / (V i ¹Uº) -

¹g¹ Gºº = ∫₀^1 ¹ G- H- Vº 4 U¹ G¸Hº dH V / (V i ¹Uº) -

and

¹maxf f ¹Gº- g¹ Gºg º = 4UG 4 k¹Vº G ∫₀^1 ¹ -H- Vº 4 U¹ ¸Hº dH V / (V i ¹Uº) ¸

4UG ∫₀^1 ¹ G- H- Vº 4 k¹Vº ¹ ¸Hº dH V / (V i ¹Uº) •

(iv) Use the identity ¹minf0- 1gº = ¹0º ¸ ¹1º ¹maxf0- 1gº to conclude, for G 2 ¹0- ¼,

X¢¹Gº = V / (V i ¹Uº) 4 UG 4 k¹Vº G ∫₀^1 ¹ G- H- Vº 4 U¹ ¸Hº dH ¸

4 k¹Vº G ∫₀^1 ¹ -H- Vº 4 U¹ ¸Hº dH ¸ ∫₀^1 ¹ G- H- Vº 4 k¹Vº ¹ ¸Hº dH !•

(v) Show by conditioning on the first event (which could correspond to either a claim arrival or to killing) that

j¹0º = V / (V ¸ _) ¸ _ / (V ¸ _) ∫₀^1 P¹ 2 dGº j¹Gº ¸ P¹ > º j¹ º •

(vi) Argue that we can write j¹ º = a j¹0º ¸ a , with

$$a := \frac{X\ {}^{1}\ {}^{o}}{1\quad X_{¸}\ {}^{1}\ {}^{o}}\text{-}\quad a\ := \frac{X_{¢}{}^{1}\ {}^{o}}{1\quad X_{¸}\ {}^{1}\ {}^{o}}\text{-}$$

and $j\ {}^{1}0^{o} = \text{b}\ j\ {}^{1}\ {}^{o}\ ¸\ \text{b}$, with

$$\text{b} := \frac{\_\int_0^1 \text{P}^{1}\quad 2\ dG^{o}\, X_{¸}\, {}^{1}G^{o}\ ¸\ \_\,\text{P}^{1}\ >\ {}^{o}}{V\ ¸\ \_\quad \_\int_0^1 \text{P}^{1}\quad 2\ dG^{o}\, X\ \ {}^{1}G^{o}}\text{-}\ \text{b}\ := \frac{V\ ¸\ \_\int_0^1 \text{P}^{1}\quad 2\ dG^{o}\, X_{¢}{}^{1}G^{o}}{V\ ¸\ \_\quad \_\int_0^1 \text{P}^{1}\quad 2\ dG^{o}\, X\ \ {}^{1}G^{o}}\text{•}$$

(vii) Conclude that for any $G\ 2\ {}^{1}0\text{-}\ \ ¼$,

$$j\ {}^{1}G^{o} = X_{¸}\,{}^{1}G^{o}\ j\ {}^{1}\ \ {}^{o}\ ¸\ X\ \ {}^{1}G^{o}\ j\ {}^{1}0^{o}\ ¸\ X_{¢}{}^{1}G^{o}\text{-}$$

with

$$j\ {}^{1}0^{o} = \frac{\text{a b}\ ¸\ \text{b}}{1\quad \text{ab}}\text{-}\qquad j\ {}^{1}\ \ {}^{o} = \frac{\text{ab}\quad ¸\ \text{a}}{1\quad \text{ab}}\text{•}$$

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Chapter 6
Level-dependent dynamics

Abstract — This chapter considers a variant of the Cramér-Lundberg model which has the special feature that the claim arrival rate $\lambda(x)$ and the premium rate $r(x)$ are functions of the current surplus level x. If only the premium rate depends on that surplus level, then duality holds with a so-called *shot-noise M/G/1 queue*, which can be used to derive the ruin probability from a known queueing result. When both rates are level-dependent, we exploit the fact that the integro-differential equation for the survival probability only involves those two rates as a fraction $r(x)/\lambda(x)$. This reduces the determination of the ruin probability to that for the case in which only the premium rate is level-dependent. Then we consider a model in which the claim interarrival times depend on the current surplus level in a specific way: they equal an exponentially distributed quantity minus a fraction of the current surplus level, truncated at 0. The chapter is concluded by the analysis of a model in which tax payments are deducted from the premium income whenever the surplus process is at a running maximum, leading to the so-called *tax identity*.

6.1 Introduction

In this chapter we consider the Cramér-Lundberg model, but now with the distinguishing feature that the claim arrival rate and the premium rate are not constant, but rather functions of the current surplus level, say $\lambda(x)$ and $r(x)$, respectively, when the surplus level is x. We throughout assume that $r(0) = 0$ and that $r(x) > 0$ for all $x > 0$. This model variant has evident applications: it can for instance be used in situations when the insurance firm adapts its premium rate to the current surplus level, but it also allows one to model the effect of receiving interest (cf. Chapter 4).

The above dynamics entail that the reserve level process obeys the integral equation

$$X_u(t) = u + \int_0^t r(X_u(s))\,\mathrm{d}s - \sum_{i=1}^{N(t)} B_i$$

where the claim arrival process $N(t)$ is such that the Poisson arrival rate at time t is $\lambda(X_u(t))$. More precisely, as $\Delta t \downarrow 0$,

$$\mathbb{P}(N(t+\Delta t) - N(t) = 1 \mid X_u(s), s \in [0,t]) = \lambda(X_u(t))\,\Delta t + o(\Delta t),$$

and

$$\mathbb{P}(N(t+\Delta t) - N(t) = 0 \mid X_u(s), s \in [0,t]) = 1 - \lambda(X_u(t))\,\Delta t + o(\Delta t),$$

with the probability of two or more claim arrivals in an interval of length Δt being $o(\Delta t)$.

The objective of this chapter is to analyze the ruin probability $\psi(u)$, i.e., the probability of $X_u(t)$ ever dropping below 0. As it turns out, for general functions $\lambda(x)$ and $r(x)$ the evaluation of the time-dependent ruin probability $\psi(u,t)$ is beyond reach, even when considering ruin before an exponentially distributed epoch. A special case of this level-dependent model has already been dealt with in Chapter 5, where the claim arrival rate and the premium rate (and actually also the claim-size distribution) depend on whether or not the current level of the surplus is above or below a given threshold v. Another special case was treated in Chapter 4, where $r(x) \equiv rx$ and $\lambda(x) \equiv \lambda$, but where two-sided jumps are allowed.

In Section 6.2, restricting ourselves to the case that only the premium rate is level-dependent (i.e., there is a constant claim arrival rate), we manage to derive an integral equation characterizing $\psi(u)$ by exploiting a duality relation with the model's M/G/1-type counterpart (cf. Section 1.2). The case in which both the premium rate and the claim arrival rate are level-dependent is considered in Section 6.3; we in particular point out how this case can be reduced to the one in which only the premium rate is level-dependent. In Section 6.4 we analyze a model that does not fit in the framework sketched above: the claim interarrival times equal an exponentially distributed quantity minus a fraction of the current surplus level, truncated at 0. Section 6.5 considers another specific level-dependent risk model, leading to the so-called *tax identity*.

6.2 Level-dependent premium rate

In this section we restrict ourselves to the case of a level-dependent premium rate $r(x)$, keeping the claim arrival rate $\lambda(x) \equiv \lambda$ fixed. We relate the risk process to a dual queueing process, and determine the steady-state workload density $f(y)$ of that queueing process. By duality, $f(u)$ (with $u \geq 0$ denoting the insurance firm's initial surplus) equals the derivative of the survival probability $\bar\psi(u) := 1 - \psi(u)$, and hence minus the derivative of the ruin probability $\psi(u)$.

▷ *Duality.* Where in Section 1.2 we introduced the dual queueing model of the conventional Cramér-Lundberg model, we now follow a similar reasoning for the Cramér-Lundberg variant with a level-dependent premium rate. To this end, consider the surplus process $X_u(s)$ for $s \in [0,t]$ as defined above, for a given horizon $t > 0$. To construct its dual queueing process, define the associated queue $Q(s)$, for $s \in [0,t]$, as follows:

Apply time reversal on the interval $[0,t]$. This concretely means that the process' jumps are now *positive*.
Apply reflection at zero (provided that $r(\cdot)$ is such that zero can be reached) to prevent the process from attaining negative values.
Start the queue with a zero workload: $Q(0) = 0$.

In this case the workload dynamics are governed by the following relation:

$$Q(t) = \sum_{i=1}^{N(t)} B_i - \int_0^t r(Q(s))\,\mathrm{d}s. \tag{6.1}$$

The claim is that the finite-time ruin probability $p(u,t)$ coincides with the probability of the workload level $Q(t)$ exceeding u, where it is a crucial element that we have that $Q(0) = 0$; likewise, the all-time ruin probability $p(u)$ matches the probability of the stationary workload level $Q(\infty)$ exceeding u, again under $Q(0) = 0$. This claim is formalized in the following theorem. Here $\tau(u)$ denotes the first time that the reserve level process $X_u(t)$ attains a negative value, i.e., the ruin time.

Theorem 6.1 *For any $t > 0$, the events $\{\tau(u) \le t\}$ and $\{Q(t) > u\}$ coincide. In particular, the events $\{\tau(u) < \infty\}$ and $\{Q(\infty) > u\}$ coincide.*

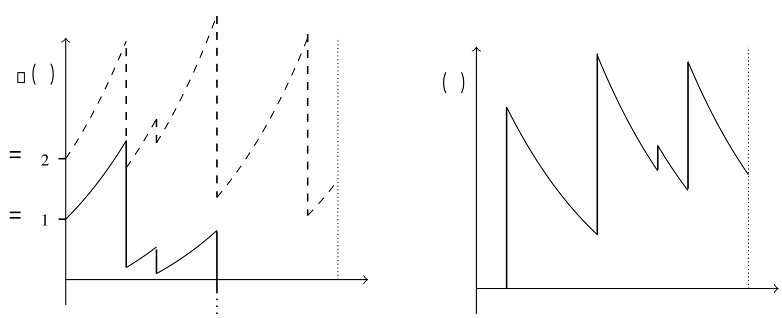

Fig. 6.1 Left panel: reserve level process $X_u(t)$ for initial surplus u_1 (solid lines) and for initial level u_2 (dashed lines). Right panel: the constructed workload process $Q(t)$, with the time-reversed arrival process.

Proof. The proof of this result relies on a sample-path comparison technique. Let there be N claims in the reserve level process $X_u(t)$ between times 0 and t (N is Poisson distributed with parameter λt); call these times t_1 up to t_N. Because of the time reversal, the jumps in the dual queueing process $Q(t)$ happen at times $v_n := t - t_{N-n+1}$, for $n = 1,\ldots,N$. The claims $B_1,\ldots,B_N$ in the reserve level process $X_u(t)$ correspond to upward jumps in the queueing process $Q(t)$ of size B_{N-i+1}.

Let the deterministic function $x_u(s)$ solve $x_u'(s) = r(x_u(s))$ under $x_u(0) = u$. Evidently, there is a monotonicity as a function of the initial surplus level: if $u_1 < u_2$, then $x_{u_1}(s) < x_{u_2}(s)$. The proof of the equivalence of the events $\{\tau(u) \le t\}$ and $\{Q(t) > u\}$ amounts to establishing two inclusions.

We first consider the scenario that $Q(t) > u$, corresponding to the path of $X_{u_1}(t)$ in the left panel of Figure 6.1 (i.e., the solid graph). Then, due to the monotonicity that we observed above,

$$Q(T_N-) = x_{Q(t)}(t_1) - B_1 > x_u(t_1) - B_1 = X_u(t_1).$$

If $Q(T_N-) = 0$, then we find that $X_u(t_1) < 0$, so that indeed $\tau(u) \le t$. If on the contrary $Q(T_N-) > 0$, we iterate the above argument to conclude that $Q(T_{N-1}-) > X_u(t_2)$:

$$\begin{aligned} Q(T_{N-1}-) &= x_{Q(T_N-)}(t_2 - t_1) - B_2 \\ &> x_{X_u(t_1)}(t_2 - t_1) - B_2 = X_u(t_2). \end{aligned}$$

Again we can consider separately the cases $Q(T_{N-1}-) = 0$ and $Q(T_{N-1}-) > 0$, where in the former case we can conclude that $X_u(t_2) < 0$ and hence $\tau(u) \le t$. Continuing along these lines, due to the fact that $Q(T_1-) = 0$, this procedure will eventually yield a T_j, with $j \in \{1, \dots, N\}$, such that $Q(T_j-) = 0$. As a consequence, for this j we have that $X_u(t_{N-j+1}) < 0$, which implies that $\tau(u) \le t$, as desired.

Conversely, now suppose that $Q(t) \le u$, corresponding to the path of $X_{u_2}(t)$ in the left panel of Figure 6.1 (i.e., the dashed graph). Then, using the monotonicity once more,

$$Q(T_N-) = x_{Q(t)}(t_1) - B_1 \le x_u(t_1) - B_1 = X_u(t_1).$$

This relation can be iterated in the same way as was done above, so as to obtain $Q(T_j-) \le X_u(t_{N-j+1})$, for all $j \in \{1, \dots, N\}$. Together with $Q(s) \ge 0$ for all $s \in [0, t]$, this implies that at all claim arrivals the reserve level process is non-negative. As ruin can only occur at claim arrivals, this means that no ruin occurs in $[0, t)$, i.e., that $\tau(u) > t$.

This completes the proof.

$\triangleright$ *The queueing workload process.* Justified by the duality result of Theorem 6.1, our objective is to describe the distribution of the stationary workload $Q(\infty)$. We use a level-crossing argument to characterize the distribution of $Q(\infty)$; cf. the approach presented in Section 1.6. Let $f(y)$ be the density of the stationary workload as a function of $y > 0$ (assumed to exist), which equals $-p'(y)$ by virtue of the above duality result. In addition, let $\pi(0)$ denote the probability that the stationary workload is 0.

For any level $x > 0$ we have the following integral equation characterizing the density $f(x)$.

Theorem 6.2 *For $x > 0$,*

$$r(x)f(x) = \lambda \int_0^x \mathbb{P}(B > x - y) f(y)\,\mathrm{d}y + \lambda\,\pi(0)\,\mathbb{P}(B > x). \tag{6.2}$$

The validity of this integral equation can be argued in the same way as we arrived at its counterpart (1.11) for the conventional M/G/1 queue; the left-hand side of (6.2) can be interpreted as the probability flux through the level x from above, and the right-hand side as the probability flux through x from below.

Remark 6.1 Whether $\pi(0) > 0$, i.e., whether the workload can become zero, depends on $r(\cdot)$. Introduce, for $0 \leqslant x < y < \infty$,

$$\tau(y,x) := \int_x^y \frac{1}{r(z)}\,\mathrm{d}z.$$

It represents the time to go from level y to level $x < y$ in the absence of arrivals. In case $\tau(y,0) < \infty$, the level 0 can be reached from level y in a finite amount of time. Recalling that $r(y) > 0$ for all $y > 0$, it follows that if $\tau(y,0) < \infty$ for some $y > 0$, then $\tau(y,0) < \infty$ for all $y > 0$. The case $r(y) = ay$ provides an example where the level 0 cannot be reached (once having deviated from it). ◊

The challenge is to compute the density $f(x)$ from the above integral equation (6.2), which is known in the literature as a *Volterra integral equation* of the second kind. We restrict ourselves to the case that $\pi(0) > 0$, i.e., level 0 can be reached. It is well-known that the solution of such an integral equation can be expressed in terms of an infinite sum. To this end, introduce the function $g(x) := \lambda\,\mathbb{P}(B > x)$ for $x \geqslant 0$ as well as the *kernel* $K(x,y) := g(x-y)/r(x)$ for $0 \leqslant y < x < \infty$. Using these objects, we arrive at the alternative representation

$$f(x) = K(x,0)\,\pi(0) + \int_0^x K(x,y) f(y)\,\mathrm{d}y. \tag{6.3}$$

Define the kernels $K_n(y,x)$ iteratively by $K_1(y,x) := K(y,x)$ and

$$K_n(y,x) := \int_x^y K_{n-1}(y,z)\,K(z,x)\,\mathrm{d}z$$

for $0 \leqslant x < y < \infty$ and $n \in \{2,3,\dots\}$. The well-known *Picard iteration* applied to (6.3) gives

$$\begin{aligned} f(x) &= K(x,0)\,\pi(0) + \int_0^x K(x,y)\left(K(y,0)\,\pi(0) + \int_0^y K(y,z) f(z)\,\mathrm{d}z\right)\mathrm{d}y \\ &= \cdots = \pi(0) \sum_{n=1}^{\infty} K_n(x,0). \end{aligned} \tag{6.4}$$

The convergence of the sum featuring in (6.4) is an immediate consequence of the following lemma, implying that $K(x,y) := \sum_{n=1}^{\infty} K_n(x,y)$ is well-defined.

Lemma 6.1 *For* $0 \leq y < x < \infty$ *and* $n \in \{1,2,\dots\}$,

$$K_n(x,y) \leq \frac{\lambda^n R(x,y)^{n-1}}{r(x)(n-1)!}.$$

Proof. We prove this by induction. For $n = 1$ the stated follows from $\lambda\bar{B}(x-y) \leq \lambda$; we thus have that $K(x,y) \leq \lambda/r(x)$. Now suppose the claim holds for $n-1$. Then, using the induction hypothesis,

$$K_n(x,y) = \int_y^x K_{n-1}(x,z)K(z,y)\,\mathrm{d}z \leq \int_y^x \frac{\lambda^{n-1}R(x,z)^{n-2}}{r(x)(n-2)!}\frac{\lambda}{r(z)}\,\mathrm{d}z. \tag{6.5}$$

Observing that

$$\frac{\mathrm{d}}{\mathrm{d}z}R(x,z) = -\frac{1}{r(z)},$$

we have that the right-hand side of (6.5) equals

$$\left[-\frac{\lambda^n R(x,z)^{n-1}}{r(x)(n-1)!}\right]_{z=y}^{x} = \frac{\lambda^n R(x,y)^{n-1}}{r(x)(n-1)!},$$

as desired.

We end up with the following result, describing the stationary workload density (and hence also minus the derivative of the ruin probability in the associated ruin model, as pointed out above). It uses the object

$$C := 1 + \int_{0+}^{\infty} K(y,0)\,\mathrm{d}y.$$

Theorem 6.3 *If* $C < \infty$, *then* $\pi(0) = 1/C$ *and, for* $y > 0$,

$$f(y) = \frac{K(y,0)}{C}.$$

▷ *Exponentially distributed claim sizes.* The integral equation that we derived generally does not allow an explicit solution. In the case that the claim sizes are exponentially distributed (with some parameter $\mu > 0$), however, the density $f(y)$ can be evaluated in closed form, as shown below.

Our starting point is the integral equation (6.2), which in this specific case takes the form

$$r(y)f(y) = \lambda\int_{0+}^{y} e^{-\mu(y-z)}f(z)\,\mathrm{d}z + \lambda\,\pi(0)\,e^{-\mu y}. \tag{6.6}$$

Multiplying the full equation by $e^{\mu y}$ and writing $\varphi(y) = r(y)f(y)e^{\mu y}$, we readily obtain, for $y > 0$,

$$¹H^o = \_ \int_0^{H} \frac{¹I^o}{A¹I^o}\, dI \;¸\; \_ \;\; ¹0^o•$$

Differentiation of this equality with respect to H yields

$$^0¹H^o = \_ \;\frac{¹H^o}{A¹H^o}•$$

This elementary differential equation can be solved in the standard manner. Indeed, for some constant and Y ¡ 0, we obtain

$$¹H^o = \;_Y \exp \;\_ \int_Y^{H} \frac{1}{A¹I^o}\, dI \;=\; _Y 4^{–\,'¹H\text{-}Y^o}•$$

We now distinguish two cases.

First consider the case that $'¹H\text{-}0^o = 1$. From the normalization equation we then find that

$$5¹H^o = \frac{4\;^{`H}}{A¹H^o}\, 4^{–\,'¹H\text{-}1^o} \int_0^{1} \frac{4\;^{`I}}{A¹I^o}\, 4^{–\,'¹I\text{-}1^o}\, dI \;• \qquad (6.7)$$

If $'¹H\text{-}0^o \;\ddot{Y}\; 1$, then level 0 can be reached in finite time and there is an atom at 0 (i.e., $¹0^o$ ¡ 0). It follows that

$$5¹H^o = \frac{4\;^{`H}}{A¹H^o}\;_0 4^{–\,'¹H\text{-}0^o}• \qquad (6.8)$$

The constants $_0$ and $¹0^o$ can be identified as follows. In the first place we have that

$$¹0^o \;¸\; _0 \int_{0¸}^{1} \frac{4\;^{`H}}{A¹H^o}\, 4^{–\,'¹H\text{-}0^o} dH = 1\text{-}$$

which follows from the normalization. In the second place,

$$_0 4^{–\,'¹H\text{-}0^o} = \_ \;\; _0 \int_{0¸}^{H} \frac{1}{A¹I^o} 4^{–\,'¹I\text{-}0^o}\, dI \;¸\;\; ¹0^o \;\text{-}$$

which follows by (6.6), implies (by inserting H # 0) that $_0 = \_ \;\; ¹0^o$. We thus conclude that

$$¹0^o = \; 1 \;¸\; \_ \int_{0¸}^{1} \frac{4\;^{`H}}{A¹H^o}\, 4^{–\,'¹H\text{-}0^o} dH \;\;^{1} •$$

Incidentally, a special case is $A¹H^o \quad A$. Now $'¹H\text{-}0^o = H•A \;\ddot{Y}\; 1$ and

$$5¹H^o = \frac{0}{A} \exp \quad \frac{=}{A} \quad ` \; H \;\text{-}$$

leading to the familiar result for the Cramér-Lundberg model with exponential claim sizes (as well as the M/M/1 queue).

In the case of exponentially distributed claim sizes and $'¹H\text{-}0^o \;\ddot{Y}\; 1$, there is an alternative approach to compute the density $f(x)$ and the atom $\bar{\psi}(0)$: the iterated kernels $K_n(w,x)$ can be explicitly evaluated, in a way that borrows elements from the proof of Lemma 6.1. We leave these computations as Exercise 6.1 to the reader.

6.3 Level-dependent premium rate and claim arrival rate

In this section we allow both the premium rate and the claim arrival rate to be level-dependent. We derive an integro-differential equation for the survival probability $\bar{\psi}(u) = 1 - \psi(u)$ corresponding to the initial surplus u. In this context the duality correspondence does not apply (see Remark 6.2 below). Instead we use the Kolmogorov forward equation method, i.e., Method 4 of Chapter 1, see also Section 4.3. Concretely, looking ahead an infinitesimal amount of time Δt, we can write

$$\bar{\psi}(u) = (1 - \lambda(u)\Delta t)\,\bar{\psi}(u + r(u)\Delta t) + \lambda(u)\Delta t \int_0^u \bar{\psi}(u-y)\,\mathbb{P}(B \in \mathrm{d}y) + o(\Delta t). \quad (6.9)$$

Bringing $\bar{\psi}(u + r(u)\Delta t)$ to the left-hand side and dividing by Δt, we obtain after letting $\Delta t \downarrow 0$ that, in self-evident notation,

$$\begin{aligned} r(u)\bar{\psi}'(u) &= \lambda(u)\bar{\psi}(u) - \lambda(u)\int_0^u \bar{\psi}(u-y)\,\mathbb{P}(B \in \mathrm{d}y) \\ &= \lambda(u)\bar{\psi}(u) + \lambda(u)\int_0^u \bar{\psi}(u-y)\,\mathrm{d}\mathbb{P}(B > y). \end{aligned} \quad (6.10)$$

Integration by parts, denoting the derivative of the survival probability $\bar{\psi}(u)$ by $f(u)$ (which hence equals minus the derivative of the ruin probability $\psi(u)$) yields the following result.

Theorem 6.4 *For $u > 0$,*

$$r(u)f(u) = \lambda(u)\int_{0+}^u \mathbb{P}(B > u - y) f(y)\,\mathrm{d}y + \lambda(u)\bar{\psi}(0)\,\mathbb{P}(B > u). \quad (6.11)$$

Remark 6.2 As mentioned above, when the arrival rate is not a constant, duality no longer holds. In the queueing setting, one can still apply a level crossing argument to arrive at the following balance equation for the steady-state queueing workload density $v(x)$, with $V(0)$ the steady-state probability of zero workload:

$$r(x)v(x) = \int_{0+}^x \lambda(y)\,\mathbb{P}(B > x - y)\,v(y)\,\mathrm{d}y + \lambda(0)V(0)\,\mathbb{P}(B > x).$$

This equation is different from Equation (6.11) for $f(x)$ derived above, unless $\lambda(x) \equiv \lambda$. ◊

Let us now turn to the solution of (6.11). Introducing $z(u) := \lambda(u)/r(u)$, assuming that $\lambda(u) > 0$ for all $u > 0$, (6.11) becomes:

$$z(u)\,f(u) = \lambda \int_0^u \mathbb{P}(B > u - v)\,f(v)\,\mathrm{d}v + \lambda\,p(0)\,\mathbb{P}(B > u). \tag{6.12}$$

This equation has the exact same structure as (6.2), and hence Theorem 6.3 provides an expression for $f(u)$.

Remark 6.3 The implication of the fact that Equation (6.11) can be rewritten such that $r(u)$ and $\lambda(u)$ appear only as a ratio, is that one can easily translate results for one set of $(r(u), \lambda(u))$ into those for another, perhaps more convenient, set. Such a translation basically amounts to simultaneously scaling space and time in the same way. ◇

6.4 A specific level-dependent model

In this section we consider a variant of the standard Cramér-Lundberg model, where a high surplus level leads to an increase in the rate at which new claims arrive.

▷ *Model and motivation.* The claim arrival process has been adapted in the following manner. Let $\sigma_1, \sigma_2, \ldots$ be a sequence of independently sampled random variables, each having an exponential distribution with parameter λ, independent of everything else. When the surplus level right after the i-th claim arrival takes some value y, then the next inter-claim time equals $\max\{0, \sigma_i - cy\}$, where c is a positive constant. Observe that when the current surplus level is high, with high likelihood the inter-claim time equals 0, thus leading to an immediate new claim arrival. This mechanism is such that when the surplus level is large, there is a cascade of claims, so that the reserve level is pulled down, whereas if the surplus level is small, the model effectively behaves as the conventional Cramér-Lundberg model. This suggests that the all-time ruin probability will be 1, as the surplus process will not drift to ∞. This property will be corroborated by the analysis below. Note that, while ruin in this model eventually occurs with probability one, the time-dependent ruin probability is strictly smaller than one.

▷ *Decomposition.* As before, our objective is to evaluate the time-dependent ruin probability over an exponentially distributed horizon, i.e., $p(u, T_\beta)$. Following the procedure of Chapter 1, we do so by evaluating the Laplace transform $\pi(\alpha, \beta)$ of $p(\cdot, T_\beta)$, defined by

$$\pi(\alpha, \beta) := \int_0^\infty e^{-\alpha u}\,p(u, T_\beta)\,\mathrm{d}u.$$

The main idea is that, by conditioning on the first claim arrival, we can express the transform $\pi(\alpha, \beta)$ in itself, but evaluated in different arguments. To this end, considering the evaluation of the probability $p(u, T_\beta)$, observe that two scenarios are relevant. In the first place, if the exponentially distributed random variable with parameter λ, say τ, is smaller than cu, then a next claim arrives instantly. This could lead to instantaneous ruin if its size is larger than u, and alternatively can bring the surplus process down to a level that lies strictly between 0 and u. In the second place, τ can be larger than cu. According to the mechanism that we introduced, then the claim arrives after $\tau - cu$ time units and, again, this can lead to either immediate ruin, or to a surplus level between 0 and u, at least if the time horizon T_β has not been exceeded.

The reasoning of the preceding paragraph can be expressed in formulas, as follows:

$$p(u,T_\beta) = p_1(u,T_\beta) + p_2(u,T_\beta).$$

Here $p_1(u,T_\beta)$ corresponds to the first scenario, i.e.,

$$p_1(u,T_\beta) = (1-e^{-\lambda cu})\left(\int_0^u p(u-v,T_\beta)\,\mathbb{P}(B\in \mathrm{d}v) + \int_u^\infty \mathbb{P}(B\in \mathrm{d}v)\right),$$

and $p_2(u,T_\beta)$ to the second scenario, i.e.,

$$p_2(u,T_\beta) = \int_{cu}^\infty \lambda e^{-\lambda s}\,\mathbb{P}(T_\beta > s-cu)\left(\int_0^{u+r(s-cu)} p(u+r(s-cu)-v,T_\beta)\,\mathbb{P}(B\in \mathrm{d}v) + \int_{u+r(s-cu)}^\infty \mathbb{P}(B\in \mathrm{d}v)\right)\mathrm{d}s.$$

Note that the probability $\mathbb{P}(T_\beta > s-cu) = e^{-\beta(s-cu)}$ in the previous display reflects the requirement that the exponential killing time should not have been exceeded at the next claim arrival time. Also, note that $u + r(s-cu)$ is the surplus level at the next claim arrival in case $\tau = s \geqslant cu$.

◦ *Evaluation of transforms.* In the sequel, we let $\pi_i(\alpha,\beta)$ be the Laplace transform of $p_i(\cdot,T_\beta)$, for $i=1,2$. Focusing on $\pi_1(\alpha,\beta)$, we first evaluate the integral

$$\int_0^\infty e^{-\alpha u}(1-e^{-\lambda cu})\int_0^u p(u-v,T_\beta)\,\mathbb{P}(B\in \mathrm{d}v)\,\mathrm{d}u.$$

Swapping the order of the integrals and recognizing the Laplace transform of a convolution, this is seen to equal

$$b(\alpha)\pi(\alpha,\beta) - b(\alpha+\lambda c)\pi(\alpha+\lambda c,\beta). \tag{6.13}$$

Along similar lines,

$$\int_0^\infty e^{-\alpha u}(1-e^{-\lambda cu})\int_u^\infty \mathbb{P}(B\in \mathrm{d}v)\,\mathrm{d}u = \frac{1-b(\alpha)}{\alpha} - \frac{1-b(\alpha+\lambda c)}{\alpha+\lambda c}. \tag{6.14}$$

We conclude that $\pi_1(\alpha,\beta)$ is the sum of (6.13) and (6.14).

Now consider the evaluation of $c_2{}^1U\text{-}\,V^{o}$. We are to calculate two triple integrals, the first one being

$$\int_0^1 4^{\,UD} \int_{2D}^1 \_4^{\,-B}\, 4^{\,V^1B\ \ 2D^o} \int_0^{D\,\text{¸}A^1B\ \ 2D^o} ?^1D \text{¸} A^1B \quad 2D^o \quad E\text{-})_V{}^o\, P^1 \quad 2\, dE^o\, dB\, dD\text{-}$$

which, by substituting $C = D$ ¸ $A^1B \quad 2D^o$, reads

$$\frac{\_}{A}\int_0^1 4^{\,UD} \int_D^1 4^{\,-^{11}C\ \ D^o\bullet A\,\text{¸}2D^o}\, 4^{\,V^1C\ \ D^o\bullet A} \int_0^C ?^1C \quad E\text{-})_V{}^o\, P^1 \quad 2\, dE^o\, dC\, dD\bullet$$

By first swapping the outer integral and the middle integral, and subsequently the inner integral and (what has become) the middle integral, we obtain

$$\frac{\_}{A}\int_0^1 \int_0^C \int_0^C 4^{\,UD} 4^{\,-^{11}C\ \ D^o\bullet A\,\text{¸}2D^o}\, 4^{\,V^1C\ \ D^o\bullet A}\, ?^1C \quad E\text{-})_V{}^o\, dD\, P^1 \quad 2\, dE^o\, dC\bullet$$

Evaluating the inner integral, and swapping the order of the two remaining integrals, we thus obtain

$$\frac{\_}{A}\int_0^1 \int_E^1 \frac{4^{\,{}^1\_\,\text{¸}\,V^oC\bullet A} \quad 4^{\,{}^1U\text{¸}\_2^oC}}{U \text{¸} \_2 \quad {}^1\_ \text{¸} V^o\bullet A}\, ?^1C \quad E\text{-})_V{}^o\, dC\, P^1 \quad 2\, dE^o\text{-}$$

which equals

$$\frac{\_}{A}\;\frac{1^{11}\_ \text{¸} V^o\bullet A^o\, c^{11}\_ \text{¸} V^o\bullet A\text{-}\,V^o \quad 1^1U \text{¸} \_2^o\, c^1U \text{¸} \_2\text{-}\,V^o}{U \text{¸} \_2 \quad {}^1\_ \text{¸} V^o\bullet A}\;\bullet \tag{6.15}$$

The second triple integral is

$$\int_0^1 4^{\,UD} \int_{2D}^1 \_4^{\,-B}\, 4^{\,V^1B\ \ 2D^o} \int_{D\,\text{¸}A^1B\ \ 2D^o}^1 P^1 \quad 2\, dE^o\, dB\, dD\bullet$$

It can be evaluated as follows. After swapping the inner and the middle integral, we rewrite it as

$$\_\int_0^1 4^{\,UD\,\text{¸}\,V2D} \int_D^1 \int_{2D}^{2D\,\text{¸}\,^1E\ \ D^o\bullet A} 4^{\,{}^1\_\,\text{¸}\,V^oB}\, dB \quad P^1 \quad 2\, dE^o\, dD\text{-}$$

which, after swapping the outer and (what has become) the middle integral, becomes

$$\_\int_0^1 \int_0^E 4^{\,UD\,\text{¸}\,V2D} \int_{2D}^{2D\,\text{¸}\,^1E\ \ D^o\bullet A} 4^{\,{}^1\_\,\text{¸}\,V^oB}\, dB \quad dD\, P^1 \quad 2\, dE^o\bullet$$

After straightforward calculations, this expression can be rewritten as

$$\frac{\_}{\_ \text{¸} V}\;\frac{1 \quad 1^1U \text{¸} \_2^o}{U \text{¸} \_2} \quad \frac{1^{11}\_ \text{¸} V^o\bullet A^o \quad 1^1U \text{¸} \_2^o}{U \text{¸} \_2 \quad {}^1\_ \text{¸} V^o\bullet A}\;\bullet \tag{6.16}$$

We conclude that $c_2{}^1U\text{-}V^o$ is the sum of the expressions in (6.15) and (6.16).

B *Determination of the transform of the ruin probability.* We have found that $c{}^1U\text{-}V^o$ can be written as the sum of the expressions in (6.13), (6.14), (6.15) and (6.16). In the right-hand side of this equation, $c{}^1U\text{-}V^o$, $c{}^1U \,¸\, \_2\text{-}V^o$ and $c{}^{11}\_ \,¸\, V^o\bullet A\text{-}V^o$ appear. We thus obtain, for easily determined functions $^1U\text{-}V^o$, $^1U\text{-}V^o$ and $^1U\text{-}V^o$, a relation of the form

$$c{}^1U\text{-}V^o = \ {}^1U\text{-}V^o\, c{}^1U \,¸\, \_2\text{-}V^o \,¸\ {}^1U\text{-}V^o \,¸\ {}^1U\text{-}V^o\, c{}^{11}\_ \,¸\, V^o\bullet A\text{-}V^o\bullet$$

One can subsequently express $c{}^1U \,¸\, \_2\text{-}V^o$ in terms of $c{}^1U \,¸\, 2\_2\text{-}V^o$, etc. By repeatedly iterating this relation, and observing that $c{}^1U \,¸\, 9\_2\text{-}V^o$! 0 for 9 ! 1, we can obtain an expression for $c{}^1U\text{-}V^o$. In this expression $\hat{}^1A^o := c{}^{11}\_ \,¸\, V^o\bullet A\text{-}V^o$ (with V kept fixed) appears. An expression for $\hat{}^1A^o$ is derived by inserting $U = U{}^1A^o := {}^1\_ \,¸\, V^o\bullet A$, and solving the resulting linear equation in $\hat{}^1A^o$. When performing these calculations, we establish the theorem below. We denote $U_9 := U \,¸\, 9\_2$ and $U_9{}^1A^o := U{}^1A^o \,¸\, 9\_2$.

Theorem 6.5 *For any $U > 0$ and V ¡ 0,*

$$c{}^1U\text{-}V^o = \ {}^1U\text{-}V^o \,¸\ {}^1U\text{-}V^o\, \hat{}^1A^o$$
$$¸ \sum_{9=1}^{\infty} \ {}^1U_9\text{-}V^o \,¸\ {}^1U_9\text{-}V^o\, \hat{}^1A^o \prod_{8=0}^{9\,1} \ {}^1U_8\text{-}V^o\text{-}$$

where, defining the empty product as 1,

$$\hat{}^1A^o = \frac{\sum_{9=0}^{1} \ {}^1U_9{}^1A^o\text{-}V^o \prod_{8=0}^{9\,1} \ {}^1U_8{}^1A^o\text{-}V^o}{1 \quad \sum_{9=0}^{1} \ {}^1U_9{}^1A^o\text{-}V^o \prod_{8=0}^{9\,1} \ {}^1U_8{}^1A^o\text{-}V^o}\bullet \tag{6.17}$$

Remark 6.4 Convergence of the infinite sum of products that occurs in the above expression for $c{}^1U\text{-}V^o$ can be readily verified. First observe that

$${}^1U\text{-}V^o = \frac{1{}^1U \,¸\, \_2^o}{1 \quad 1{}^1U^o} \frac{U \,¸\, \_2 \quad V\bullet A}{U \,¸\, \_2 \quad {}^1\_ \,¸\, V^o\bullet A}\text{-}$$

so that ${}^1U_9\text{-}V^o$ tends to zero as $1{}^1U \,¸\, 9\_2^o$ for 9 ! 1. Secondly, notice that ${}^1U_9\text{-}V^o$ and ${}^1U_9\text{-}V^o$ tend to zero like $1\bullet 9$. Finally, use the alternating series test, also called Leibniz criterion. ^

6.5 A tax identity

In this section we consider a specific level-dependent risk model: the Cramér-Lundberg model with the additional assumption that *tax payments* are deducted from the premium income, at a constant proportion W Ÿ 1, whenever the surplus is at a running maximum. The original Cramér-Lundberg surplus process

$$X_u(t) = u + rt - \sum_{i=1}^{N(t)} X_i$$

with all-time ruin probability $\psi(u)$ and survival probability $\bar\psi(u) = 1 - \psi(u)$, is thus modified to a process $X_u(t \mid \gamma)$ (cf. Figure 6.2), with all-time ruin probability $\psi_\gamma(u)$ and survival probability $\bar\psi_\gamma(u)$. Consideration of a tax scheme of the above form is motivated by its realistic feature of carried forward losses: occurred losses can be deducted from later income and hence they reduce the taxable profit of the business.

The main result of this section is the following, remarkably simple, *tax identity*.

Theorem 6.6 *For any $u > 0$,*

$$\bar\psi_\gamma(u) = (\bar\psi(u))^{1/(1-\gamma)}. \tag{6.18}$$

The proof of the tax identity will exploit, once more, the close relationship between the Cramér-Lundberg model and the M/G/1 queue. Before proving the theorem, we first establish an auxiliary result that relates $\bar\psi(u)$ and the probability $\mathbb{P}(W_{\max} > u)$, where $W_{\max}$ denotes the maximum workload in a busy period of the M/G/1 queue with arrival rate λ and generic service time B.

Lemma 6.2 *Under the net profit condition $r > \lambda\mathbb{E}B$ we have for every $u \geq 0$:*

$$\mathbb{P}(W_{\max} > u) = \frac{r}{\lambda}\frac{\mathrm{d}}{\mathrm{d}u}\log\bar\psi(u). \tag{6.19}$$

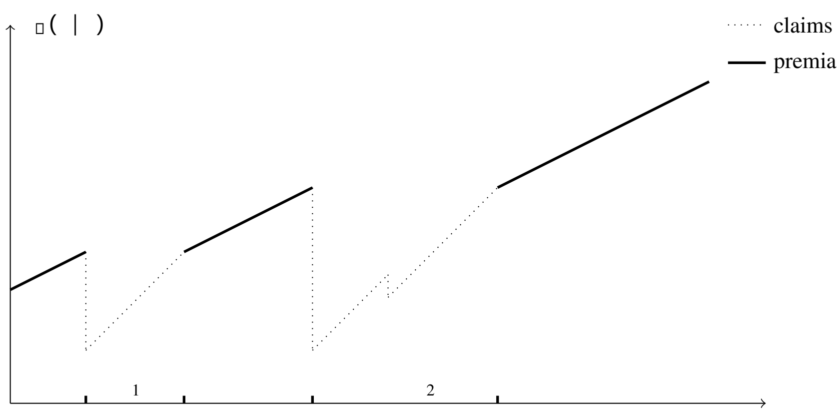

Fig. 6.2 Sample path of the Cramér-Lundberg process with tax payments.

Proof. It will be convenient to rescale the process $X_u(t)$ to a process

$$X_u'(t) := u + t - \sum_{i=1}^{N'(t)} B_i,$$

where $N'(t)$ is a homogeneous Poisson process with rate λ/r. Clearly, the survival probabilities for $X_u(t)$ and $X_u'(t)$ coincide. $X_u'(t)$ can only survive if after each claim that occurs at some running maximum $s > u$, the level s will be reached again before ruin occurs. By turning Figure 6.2 upside down and focusing on the intervals between running maxima (i.e., I_1 and I_2 in the figure), this is readily seen to be equivalent to the statement that $Q_{\max}$ does not exceed s (note that the net profit condition $r > \lambda\,\mathbb{E}B$ in the risk model corresponds to the stability condition $\lambda\,\mathbb{E}B/r < 1$ in the associated M/G/1 queue). In determining the survival probability, such 'surviving' excursions can be removed. The only claims that eventually lead to ruin are those in which ruin occurs before the process gets back to level s. Such claims occur at level s with rate

$$\lambda(s) := \frac{\lambda}{r}\,\mathbb{P}(Q_{\max} > s).$$

Hence we can interpret $\varphi(u)$ as the probability of *zero events* during (u, ∞) of a nonhomogeneous Poisson process with rate $\lambda(s)$; this constitutes a thinning of the original Poisson process with rate λ/r. We thus conclude that

$$\varphi(u) = \exp\left(-\int_u^\infty \lambda(s)\,\mathrm{d}s\right) = \exp\left(-\frac{\lambda}{r}\int_u^\infty \mathbb{P}(Q_{\max} > s)\,\mathrm{d}s\right).$$

The lemma follows by taking logarithms.

Remark 6.5 A well-known identity for the M/G/1 queue (cf. [12, p. 618]) is

$$\mathbb{P}(Q_{\max} > u) = \frac{r}{\lambda}\frac{\mathrm{d}}{\mathrm{d}u}\log \mathbb{P}(Q(\infty) < u),$$

where $Q(\infty)$ denotes the steady-state workload in the M/G/1 queue. The duality result $\varphi(u) = \mathbb{P}(Q(\infty) < u)$ (cf. Theorem 6.1), in combination with Lemma 6.2, immediately proves that identity. $\diamond$

Proof (of Theorem 6.6). We can repeat the arguments from the proof of Lemma 6.2 for the process $X_u(t \mid \gamma)$, with as only difference that the rate of increase in the running maximum is $r(1-\gamma)$ instead of r, which means that the rescaling of time leads to a Poisson intensity $\lambda/(r(1-\gamma))$ instead of λ/r. The excursions from the maximum, that are cut out, are identical to those without tax. We thus may conclude that the survival probability $\varphi_\gamma(u)$ is the probability of zero events during (u, ∞) of an inhomogeneous Poisson process with rate

$$\lambda_\gamma(s) = \frac{\lambda}{r(1-\gamma)}\,\mathbb{P}(Q_{\max} > s).$$

Hence

$$\phi_\gamma(u) = \exp\left(-\frac{\lambda}{c(1-\gamma)}\int_u^\infty P(\&_{\max} - x)\,dx\right) = (\phi(u))^{1/(1-\gamma)}$$

for any $u > 0$.

Remark 6.6 The reasoning in the above proof of Theorem 6.6 reveals that the tax identity (6.18) can be generalized to the case of an arbitrary surplus-dependent tax rate $\gamma(x)$, with $0 \le \gamma(x) < 1$, where x denotes the current surplus level. The corresponding survival probability $\phi_\gamma(u)$ is now the probability of having zero events during (u, ∞) of the inhomogeneous Poisson process with rate

$$\lambda_\gamma(x) := \frac{\lambda}{c(1-\gamma(x))} P(\&_{\max} - x),$$

and hence, by Lemma 6.2,

$$\begin{aligned}\phi_\gamma(u) &= \exp\left(-\frac{\lambda}{c}\int_u^\infty \frac{P(\&_{\max} - x)}{1-\gamma(x)}\,dx\right)\\ &= \exp\left(-\int_u^\infty \frac{d}{dx}\log\phi(x)\,\frac{1}{1-\gamma(x)}\,dx\right)\end{aligned}$$

for any $u > 0$. ◻

6.6 Discussion and bibliographical notes

Our proof of the duality result of Theorem 6.1 was based on the one of [3, Theorem III.2.1], but see also [6, Theorem 3]. A first proof of duality in the case of a level-dependent premium rate (in the insurance model) and service rate (in its queuing counterpart) was provided in [16] for the infinite-horizon case and in [4] for the finite-horizon case.

A detailed literature review on queueing models with level-dependent arrival and service rates can be found in [7, Section 3]; we restrict ourselves here to a few of the most relevant references. An influential early paper on this type of storage systems with content-dependent release rate is [14], constructing the Kolmogorov forward equations for several model variants. One of these variants concerns the model in which $c(y)$ is c for y below some threshold, and c_* else; cf. the model discussed in Chapter 5. A second variant was the shot-noise case $c(y) = cy$, which was treated in e.g. [18]. The papers [11, 20, 21] are early references on storage processes satisfying (6.1), e.g. focusing on proving the existence of solutions. The results presented in our Section 6.2 directly relate to those in [15]; see also the extensions and variants in [10, 17, 19]. An overview of level-dependent risk processes, also including some guidelines related to the numerical evaluation, can be found in [3, Ch. VIII]. A specific model that is covered by the results presented in the first three sections of this chapter, is that of 'level-dependent premium', i.e., a surplus process with liquid reserves [13].

The results presented in Section 6.4, which have appeared in [8], have a structure that resembles the one found in e.g. [5, 9]. In these papers, models are studied in which the workload process at arrival times is an autoregressive process reflected at 0, which can be seen as dual of the Cramér-Lundberg variant that we analyzed.

The tax identity in Section 6.5 was first discovered in [2]. The short proof exploiting a relation to the M/G/1 queue has been developed in [1].

Exercises

6.1 Derive the density 5¹Hº as obtained in (6.8) by explicitly determining ¢¹G-Hº. (*Hint*: Prove by induction that

$$=¹G-Hº = \frac{\_ = ' ¹G-Hº = 1}{A¹Gº¹= \quad 1º!} \, 4 \; `¹G \; Hº -$$

for = 2 f1-2-•••g.)

6.2 Consider the integral equation (6.6) for the case of exponentially distributed claim sizes, and assume that A¹Hº = AH. Check that, in this case, ¹0º = 0 and show that 5¹Hº is a Gamma density. Try to use each of the following two approaches: (i) rewrite (6.6) into a first-order differential equation; (ii) take Laplace transforms in (6.6).

6.3 Consider two queueing or risk systems as considered in Section 6.3, with arrival rate $_{8}$¹ º and premium rate $A_8$¹ º in system 8, 8 = 1-2. Assume that, for all G ¡ 0,

$$\frac{\_1 ¹Gº}{A_1 ¹Gº} = \frac{\_2 ¹Gº}{A_2 ¹Gº} •$$

Prove that the ratio of the two workload densities (or minus the derivatives of the ruin probabilities) satisfies, for all D ¡ 0,

$$\frac{5_1 ¹Dº}{5_2 ¹Dº} = \frac{A_2 ¹Dº}{A_1 ¹Dº} -$$

with some constant.

6.4 Prove that ^¹Aº is given by (6.17).

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Chapter 7
Multivariate ruin

Abstract — In this chapter we focus on a multivariate variant of the conventional Cramér-Lundberg model. Imposing an ordering condition on the individual net cumulative claim processes, it turns out that the distribution of the joint running maximum can be derived, which can be used to evaluate ruin probabilities in a multivariate context. We start by analyzing the bivariate case, to then extend the reasoning to the higher-dimensional setting. The method relies upon the Kolmogorov forward equations underlying the associated queueing process. The solution reveals a so-called quasi-product form structure. We also point out how the results from this section can be translated into corresponding results for tandem queueing networks. We conclude the chapter by deriving the corresponding multivariate Gerber-Shiu metrics (including ruin times, undershoots, and overshoots).

7.1 Introduction

While the existing ruin theory literature primarily considers the univariate setting featuring a single reserve process, in practice the position of an insurance firm is often described by multiple, typically correlated, reserve processes. This chapter therefore focuses on the evaluation of ruin probabilities in a setting with multiple correlated reserve processes.

Multivariate ruin turns out to be a challenging topic that can be dealt with explicitly only under additional assumptions. Most notably, a certain ordering between the individual net cumulative claim processes, say $\boldsymbol{Y}(t) = (Y_1(t), \ldots, Y_d(t))$ for some $d \in \mathbb{N}$, needs to be imposed. In Section 7.2 we start by dealing with the case of two net cumulative claim processes, extending this in Section 7.3 to the higher-dimensional setting. We in addition point out, in Section 7.4, how these results relate to the workload distribution in corresponding tandem queueing networks. Multivariate Gerber-Shiu metrics (including ruin times, undershoots, and overshoots) are derived in Section 7.5.

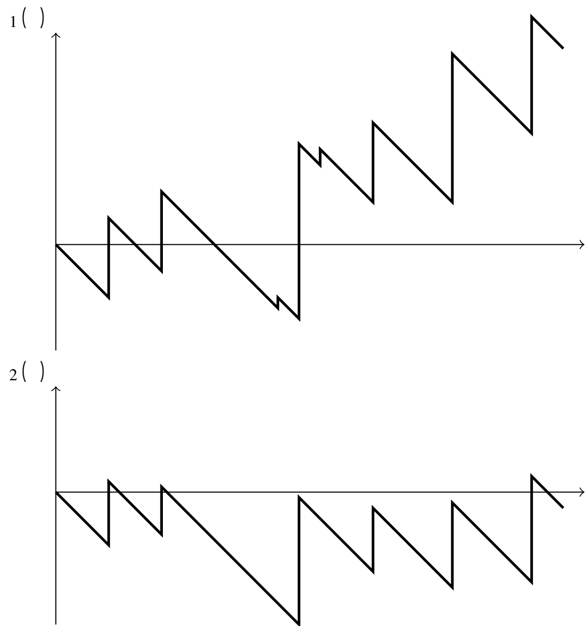

Fig. 7.1 Net cumulative claim processes $X_1(t)$ and $X_2(t)$. Observe that the processes are ordered; all jumps in $X_1(t)$ correspond to simultaneous jumps of at most that size (possibly zero) in $X_2(t)$.

7.2 Two-dimensional case

Consider the situation of two net cumulative claim processes, say $X_1(t)$ and $X_2(t)$, in which claims arrive simultaneously, according to a Poisson process with rate λ. These claims $\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots$ are 2-dimensional, componentwise non-negative i.i.d. random vectors, distributed as the generic random vector $\boldsymbol{B}$. Their entries are *ordered* in the sense that

$$\mathbb{P}\left(B^{(1)} \geq B^{(2)}\right) = 1, \tag{7.1}$$

where $B^{(i)}$ is a generic claim size corresponding to the i-th net cumulative claim process. The premium rate is r for both individual net cumulative claim processes. The bivariate Laplace exponent is therefore given by

$$\varphi(\boldsymbol{\alpha}) := \log \mathbb{E}\, e^{-\boldsymbol{\alpha}^\top \boldsymbol{X}(1)} = r\, \mathbf{1}^\top \boldsymbol{\alpha} - \lambda(1 - b(\boldsymbol{\alpha})), \tag{7.2}$$

with $b(\boldsymbol{\alpha})$ the bivariate LST corresponding to the random vector $\boldsymbol{B}$. In the setting presented in this section, we assume the per-component claim size distributions to be ordered almost surely (by the condition displayed in (7.1)), whereas the premium rates of the components are assumed to coincide. We can slightly deviate from this, as will be discussed in Remark 7.1 below. We refer to Figure 7.1 for an illustration;

observe that jumps in the process $Y_1(t)$ can correspond to jumps of size 0 in the process $Y_2(t)$.

B *Introduction of the key objects.* Our approach will rely on Method 4, as was discussed in Section 1.6. This means that we set up the Kolmogorov forward equations for the bivariate queueing process $\boldsymbol{Q}(t)$ (with $\boldsymbol{Q}(0) = \boldsymbol{0}$) that is the dual of the risk model that we defined above. The following lemma is very useful, in that it shows that ruin in the bivariate risk model (with initial capitals u_1 and u_2) can be expressed in terms of exceedance probabilities (over levels u_1 and u_2) in the bivariate dual queueing model; cf. Theorem 6.1. The lemma, that can be extended to dimensions higher than 2 in a straightforward fashion, justifies that in the sequel we focus on the queueing model only. It describes four events: ruin for both firms, ruin only for the first firm, ruin only for the second firm, and no ruin. Define, for $u > 0$ and $i = 1, 2$,

$$\tau_i(u) := \inf\{t > 0 : Y_i(t) > u\}.$$

Lemma 7.1 *For any $t > 0$,*

the events $\{\tau_1(u_1) \leqslant t, \tau_2(u_2) \leqslant t\}$ *and* $\{Q_1(t) > u_1, Q_2(t) > u_2\}$ *coincide.*
the events $\{\tau_1(u_1) > t, \tau_2(u_2) > t\}$ *and* $\{Q_1(t) \leqslant u_1, Q_2(t) \leqslant u_2\}$ *coincide.*
the events $\{\tau_1(u_1) \leqslant t, \tau_2(u_2) > t\}$ *and* $\{Q_1(t) > u_1, Q_2(t) \leqslant u_2\}$ *coincide.*
the events $\{\tau_1(u_1) > t, \tau_2(u_2) \leqslant t\}$ *and* $\{Q_1(t) \leqslant u_1, Q_2(t) > u_2\}$ *coincide.*

Proof. First observe that, based on Theorem 6.1, it follows that the events $\{\tau_i(u) \leqslant t\}$ and $\{Q_i(t) > u\}$ coincide, for $i = 1, 2$. This directly implies the first and second claim. The third claim follows from

$$\{\tau_1(u_1) \leqslant t, \tau_2(u_2) > t\} = \{\tau_1(u_1) \leqslant t\} \setminus \{\tau_1(u_1) \leqslant t, \tau_2(u_2) \leqslant t\},$$

in combination with the first claim and the fact that the events $\{\tau_1(u) \leqslant t\}$ and $\{Q_1(t) > u\}$ coincide. The fourth claim follows in exactly the same way as the third claim.

We define a few auxiliary functions. In the first place, our objective is to characterize the transform

$$\varphi_t(\boldsymbol{\alpha}) := \mathbb{E}\, e^{-\boldsymbol{\alpha}^{\rm T} \boldsymbol{Q}(t)}.$$

As before, we settle for this object evaluated at an exponentially distributed time T_q, for some killing rate q. Observe that both individual queues are of the M/G/1 type, and can therefore be analyzed relying on the techniques explained in Chapter 1, but that the challenge lies in revealing their *joint* workload distribution.

As the queueing dynamics in the interior of the positive quadrant differ from those at the boundaries, we have to introduce a few more quantities. For this reason, the object

$$\bar\varphi_t(\boldsymbol{\alpha}) := \mathbb{E}\, e^{-\boldsymbol{\alpha}^{\rm T} \boldsymbol{Q}(t)} 1\{\boldsymbol{Q}(t) > \boldsymbol{0}\}$$

plays a pivotal role in the analysis; here the (strict) inequality $\boldsymbol{Q}(t) > \boldsymbol{0}$ is to be understood in the componentwise manner, in that both $Q_1(t)$ and $Q_2(t)$ should be strictly positive. It is immediate that $W_1(t) \geq W_2(t)$ almost surely, by virtue of (7.1), so that we cannot have that $W_2(t) > 0$ while $W_1(t) = 0$. As a consequence, the following two quantities are relevant:

$$\begin{aligned}\xi_t^{(1)}(\alpha_1) &:= \mathbb{E}\, e^{-\boldsymbol{\alpha}^\top W(t)} 1\{W_1(t) > 0, W_2(t) = 0\}\\ &= \mathbb{E}\, e^{-\alpha_1 W_1(t)} 1\{W_1(t) > 0, W_2(t) = 0\}\end{aligned} \tag{7.3}$$

and

$$\begin{aligned}q_t &:= \mathbb{E}\, e^{-\boldsymbol{\alpha}^\top W(t)} 1\{W_1(t) = W_2(t) = 0\}\\ &= \mathbb{P}(W_1(t) = W_2(t) = 0) = \mathbb{P}(W_1(t) = 0).\end{aligned} \tag{7.4}$$

Note that we already have identified an expression for q_{T_β} in Section 1.6: with $\psi_1(\beta)$ the right-inverse of $\varphi(\alpha_1, 0)$, we have that, for $\beta > 0$,

$$q_{T_\beta} = \frac{\beta}{r\psi_1(\beta)}. \tag{7.5}$$

The above transforms can be translated into transforms related to ruin probabilities, as follows. Define the bivariate time-dependent ruin probability by

$$p(\boldsymbol{u}, t) := \mathbb{P}(\tau_1(u_1) \leq t, \tau_2(u_2) \leq t),$$

if the initial reserve levels of the two firms are u_1 and u_2, respectively. In addition, we denote by $p_i(u, t)$ the time-dependent ruin probability of firm i given an initial surplus u, for $i = 1, 2$. We also introduce

$$\pi(\boldsymbol{\alpha}, \beta) := \int_0^\infty\!\!\int_0^\infty e^{-\boldsymbol{\alpha}^\top \boldsymbol{u}} p(\boldsymbol{u}, T_\beta)\, \mathrm{d}u_1\, \mathrm{d}u_2, \quad \pi_i(\alpha, \beta) := \int_0^\infty e^{-\alpha u} p_i(u, T_\beta)\, \mathrm{d}u.$$

Following the line of reasoning of Remark 1.2,

$$\xi_{T_\beta}(\boldsymbol{\alpha}) = 1 - \alpha_1 \pi_1(\alpha_1, \beta) - \alpha_2 \pi_2(\alpha_2, \beta) + \alpha_1\alpha_2\, \pi(\boldsymbol{\alpha}, \beta); \tag{7.6}$$

see Exercise 7.1. As we know from Chapter 1 how to compute $\pi_i(\alpha, \beta)$ for $i = 1, 2$, it follows that in order to determine $\pi(\boldsymbol{\alpha}, \beta)$ it suffices to find $\xi_{T_\beta}(\boldsymbol{\alpha})$.

◦ *Setting up the Kolmogorov forward equations.* We mimic the argumentation that has been used in Section 1.6 to analyze the single-dimensional setting, in that we relate the relevant quantities at time $t + \Delta t$ to their counterparts at time t. Concretely, we obtain, up to $o(\Delta t)$-terms,

$$\begin{aligned}\xi_{t+\Delta t}(\boldsymbol{\alpha}) + \xi^{(1)}_{t+\Delta t}(\alpha_1) + q_{t+\Delta t} &= \xi_{t+\Delta t}(\boldsymbol{\alpha})\\ &= \xi_t(\boldsymbol{\alpha})\,\big(1 - \lambda\Delta t + \lambda\Delta t\, b(\boldsymbol{\alpha}) + r\,\mathbf{1}^\top\boldsymbol{\alpha}\,\Delta t\big) +\\ &\quad\ \xi^{(1)}_t(\alpha_1)\,\big(1 - \lambda\Delta t + \lambda\Delta t\, b(\boldsymbol{\alpha}) + r\alpha_1\,\Delta t\big) +\\ &\quad\ q_t\,\big(1 - \lambda\Delta t + \lambda\Delta t\, b(\boldsymbol{\alpha})\big).\end{aligned}$$

Recalling the definition of $i\,{}^{1}\alpha^{\circ}$, we thus obtain, in the standard manner, the following differential equation.

Lemma 7.2 *For any* $\alpha > \mathbf{0}$ *and* C ¡ 0,

$$\frac{m}{mC}\wedge_C{}^1\alpha^{\circ} \,¸\, \frac{m}{mC}\wedge_C^{{}^1 1^{\circ}}{}^1U_1{}^{\circ} \,¸\, \frac{m}{mC}@_C$$
$$= i\,{}^1\alpha^{\circ}\wedge_C{}^1\alpha^{\circ} \;¸\; i\,{}^1\alpha^{\circ} \quad AU_2 \;\wedge_C^{{}^1 1^{\circ}}{}^1U_1{}^{\circ} \;¸\; i\,{}^1\alpha^{\circ} \quad A\mathbf{1}^{>}\alpha \quad @_C•$$

B *Derivation of the transform.* The next step is to multiply the full differential equation appearing in Lemma 7.2 by the density $V4^{\;VC}$, and then integrate over $C > 0$, so as to obtain insight into the transforms at the exponentially distributed time $)_V$. Indeed, by a direct application of the identity (1.9), and realizing that

$$\wedge_0{}^1\alpha^{\circ} = \wedge_0^{{}^1 1^{\circ}}{}^1U_1{}^{\circ} = 0-$$

we find

$$V\;\wedge_{)_V}{}^1\alpha^{\circ} \,¸\, \wedge_{)_V}^{{}^1 1^{\circ}}{}^1U_1{}^{\circ} \,¸\, @_{)_V} \quad 1$$
$$= i\,{}^1\alpha^{\circ}\wedge_{)_V}{}^1\alpha^{\circ} \;¸\; i\,{}^1\alpha^{\circ} \quad AU_2 \;\wedge_{)_V}^{{}^1 1^{\circ}}{}^1U_1{}^{\circ} \;¸\; i\,{}^1\alpha^{\circ} \quad A\mathbf{1}^{>}\alpha \quad @_{)_V}•$$

After some elementary rearranging we arrive at the following expression for the transform $\wedge_{)_V}{}^1\alpha^{\circ}$:

$$\wedge_{)_V}{}^1\alpha^{\circ} = \frac{i\,{}^1\alpha^{\circ} \quad AU_2 \quad V\;\wedge_{)_V}^{{}^1 1^{\circ}}{}^1U_1{}^{\circ} \;¸\; i\,{}^1\alpha^{\circ} \quad A\mathbf{1}^{>}\alpha \quad V\;@_{)_V} \;¸\; V}{i\,{}^1\alpha^{\circ} \quad V}-$$

so that we end up with

$$\wedge_{)_V}{}^1\alpha^{\circ} = \frac{AU_2\,\wedge_{)_V}^{{}^1 1^{\circ}}{}^1U_1{}^{\circ} \;¸\; A\mathbf{1}^{>}\alpha\, @_{)_V} \quad V}{i\,{}^1\alpha^{\circ} \quad V}• \tag{7.7}$$

Inspecting the quantities appearing in the right-hand side of the previous display, recall that we have (by Equation (7.5)) an expression for $@_C$ for $C =)_V$, but note that we lack an expression for

$$\wedge_{)_V}^{{}^1 1^{\circ}}{}^1U_1{}^{\circ} = \mathrm{E}\,4^{\;U_1 \&_1{}^1)_V{}^{\circ}}\,1\mathrm{f}\&_1{}^1)_V{}^{\circ} \;¡\; 0\text{-}\&_2{}^1)_V{}^{\circ} = 0\mathrm{g}•$$

We now point out how to identify the latter quantity, for any $U_1 > 0$ and V ¡ 0.

Our strategy is to exploit that any zero of the denominator of (7.7) for which $\wedge_{)\,V}{}^1\alpha^{\circ}$ should be finite is necessarily also a zero of the numerator. Observe that we can rewrite $i\,{}^1\alpha^{\circ} \quad V = 0$ as

$$\_1{}^1\alpha^{\circ} = 2{}^1\alpha^{\circ} := \_ \quad A\mathbf{1}^{>}\alpha \;¸\; V• \tag{7.8}$$

Fixing α_1 with Re $\alpha_1 > 0$ and q, due to the lemma below we can identify a unique α_2 such that $k(\alpha) - q = 0$ while $\frac{\partial}{\partial q}k(\alpha)$ should be finite; we denote this α_2 by $\psi_2(\alpha_1, q)$.

Lemma 7.3 *For every α_1 with* Re $\alpha_1 > 0$ *and* $q > 0$, *there exists a unique* $\alpha_2 = \psi_2(\alpha_1, q)$ *with* Re $\psi_2(\alpha_1, q) >$ Re $(-\alpha_1)$ *that satisfies Equation* (7.8). *For any* $q > 0$, *the function* $\alpha_1 \mapsto \psi_2(\alpha_1, q)$ *is analytic in* Re $\alpha_1 > 0$.

Proof. We prove this by applying Rouché's theorem (see Theorem A.1). To this end, consider the contour $\mathcal{C}$ consisting of (i) the line segment

$$\mathcal{C}_I := \{-\alpha_1 + iy \mid y \in [-\rho, \rho]\},$$

with $\rho > (2\lambda + q)/c$, and (ii) to its right the semicircle

$$\mathcal{C}_B := \{-\alpha_1 + \rho e^{i\theta} \mid \theta \in [-\pi/2, \pi/2]\}.$$

The next step is to show that on this contour

$$|f_1(\alpha)| < |f_2(\alpha)|; \tag{7.9}$$

cf. Equation (7.8). First observe that we can write

$$f_1(\alpha) = \lambda\, \mathbb{E}\exp\left(-\alpha_1\left(B^{(1)} - B^{(2)}\right) - (\alpha_1 + \alpha_2)B^{(2)}\right). \tag{7.10}$$

For all α such that Re $\alpha_2 >$ Re $(-\alpha_1)$ we have that

$$|f_1(\alpha)| \leq \lambda\, \mathbb{E}\exp\left(-\mathrm{Re}\,\alpha_1\left(B^{(1)} - B^{(2)}\right) - \mathrm{Re}(\alpha_1 + \alpha_2)B^{(2)}\right) \leq \lambda,$$

recalling that we have assumed in (7.1) that $B^{(1)} \geq B^{(2)}$ almost surely. To prove (7.9) it therefore suffices to show that $|f_2(\alpha)| > \lambda$. This is done as follows.

- Observe that on the line segment $\mathcal{C}_I$ we have that $f_2(\alpha) = \lambda + q - c\,iy$. This implies that $|f_2(\alpha)| > \lambda$ on $\mathcal{C}_I$.
- Observe that on the semicircle $\mathcal{C}_B$ we have that $f_2(\alpha) = \lambda + q - c\rho e^{i\theta}$. As a consequence, $\rho > (2\lambda + q)/c$ implies that $|f_2(\alpha)| > \lambda$ on $\mathcal{C}_B$.

Now that we know that $|f_1(\alpha)| < |f_2(\alpha)|$ on $\mathcal{C}$, it follows by Rouché's theorem that the equation $f_1(\alpha) = f_2(\alpha)$ has, for a given α_1 with Re $\alpha_1 > 0$ and $q > 0$, a unique solution $\alpha_2 = \psi_2(\alpha_1, q)$ with Re $\psi_2(\alpha_1, q) >$ Re $(-\alpha_1)$; here use that the linear function $f_2(\alpha)$ has a unique zero inside $\mathcal{C}$. The first claim of the lemma now follows by sending ρ to ∞, and observing that $f_1(\alpha)$, as given by (7.10), and $f_2(\alpha)$ are analytic on and inside $\mathcal{C}$. The fact that $\psi_2(\alpha_1, q)$ is analytic in Re $\alpha_1 > 0$ follows by using the implicit function theorem; cf. [11, p. 101].

With Lemma 7.3 at our disposal and recalling Equation (7.5), by equating the numerator of (7.7) to 0, we conclude that

$$A\,l_2{}^1U_1\text{-}\,V^{o}\,\kappa^{1o}_{)V}{}^1U_1{}^{o}\ \text{¸}\ U_1\ \text{¸}\ l_2{}^1U_1\text{-}\,V^{o}\ \frac{V}{k_1{}^1V^{o}}\qquad V=0\text{-}$$

or equivalently

$$\kappa^{1o}_{)V}{}^1U_1{}^{o}=\frac{V}{A\,l_2{}^1U_1\text{-}\,V^{o}}\qquad\frac{U_1}{A\,l_2{}^1U_1\text{-}\,V^{o}}\ \text{¸}\ \frac{1}{A}\ \ \frac{V}{k_1{}^1V^{o}}\bullet$$

This can now be inserted into Equation (7.7), so as to establish an expression for the quantity of our interest, $\hat{}_{)V}{}^1\alpha^{o}$:

$$\hat{}_{)V}{}^1\alpha^{o}=\frac{1}{i\,{}^1\alpha^{o}\quad V}\ \ \frac{VU_2}{l_2{}^1U_1\text{-}\,V^{o}}\qquad\frac{U_1U_2}{l_2{}^1U_1\text{-}\,V^{o}}\ \text{¸}\ U_2\ \ \frac{V}{k_1{}^1V^{o}}\ \text{¸}\ \frac{{}^1U_1\ \text{¸}\ U_2{}^{o}V}{k_1{}^1V^{o}}\qquad V\ \bullet$$

After rearranging these terms we find the following result.

Theorem 7.1 *For any $\alpha>\mathbf{0}$ and $V\ ¡\ 0$,*

$$\hat{}_{)V}{}^1\alpha^{o}=\frac{U_1\quad k_1{}^1V^{o}}{i\,{}^1\alpha^{o}\quad V}\,\frac{V}{k_1{}^1V^{o}}\,\frac{l_2{}^1U_1\text{-}\,V^{o}\quad U_2}{l_2{}^1U_1\text{-}\,V^{o}}\bullet$$

We once more emphasize that the two-dimensional Laplace transform of the probability that both firms get ruined before $)_V$ follows from this result for $\hat{}_{)V}{}^1\alpha^{o}$; cf. (7.6).

Remark 7.1 In the above setup, we have assumed that the two premium rates, corresponding to the individual net cumulative claim processes, are identical. This we can deviate from, however, when adapting the condition given in Equation (7.1). This can be seen as follows.

Suppose that the premium rate for the 8-th net cumulative claim process is A_8, for $8=1\text{-}2$. In this case, the object we would like to evaluate is

$$\mathbb{E}\exp\left(U_1\sup_{C2»0\text{-})_V¼}\sum_{8=1}^{\#\check{O}Co}{}^{1}_{8}{}^{1o}\quad A_1C\qquad U_2\sup_{C2»0\text{-})_V¼}\sum_{8=1}^{\#\check{O}Co}{}^{12o}_{8}\quad A_2C\right)\text{-}$$

which can be rewritten as

$$\mathbb{E}\exp\left(\frac{U_1A_1}{A}\sup_{C2»0\text{-})_V¼}\sum_{8=1}^{\#\check{O}Co}\frac{A}{A_1}\,{}^{11o}_{8}\quad AC\qquad\frac{U_2A_2}{A}\sup_{C2»0\text{-})_V¼}\sum_{8=1}^{\#\check{O}Co}\frac{A}{A_2}\,{}^{12o}_{8}\quad AC\right)\text{-}$$

which we can evaluate relying on the material presented in this section, under the proviso that the assumption displayed in Equation (7.1) is replaced by

$$\mathbb{P}\left({}^{11o}\bullet A_1>{}^{12o}\bullet A_2\right)=1\bullet$$

This observation will be used when discussing tandem systems in Section 7.4. ^

Remark 7.2 Consider the system in stationarity, under the net-profit condition $\_\,\mathbb{E}\ {}^{11o}\ \ddot{Y}\ A$ (cf. Remark 1.1). Then the joint LST of the two queue workloads can be found by letting $V \# 0$. This requires the evaluation of the zero $I_2{}^1U_1\text{-}0^o$, which [6, Section 3] has an interpretation in terms of the LST $*{}^1U_1{}^o$ of the extra workload accumulated in the first queue at the end of a busy period of the second queue:

$$*{}^1U_1{}^o = 1 \quad \frac{U_1 \text{ ¸ } I_2{}^1U_1\text{-}0^o}{\_}\bullet$$

In [6] this observation was used to conclude that the following decomposition holds:

$$^1\&_1\text{-}\&_2{}^o \overset{\mathrm{d}}{=} {}^1\tilde{\&}_1\text{-}\&_2{}^o \text{ ¸ } {}^1\hat{\&}_1\text{-}0^o\text{-} \tag{7.11}$$

where the two vectors in the right-hand side are independent. Here $\tilde{\&}_1$ denotes the stationary workload in the first queue when the extra workload in this queue is removed at the end of each busy period of the second queue, and $\hat{\&}_1$ denotes the stationary workload in an M/G/1 queue with arrival rate $\_$ and service requirements having LST $*{}^1U_1{}^o$. ^

7.3 Higher-dimensional case

We proceed by recursively solving the case of $3 \ 2 \ \mathrm{f}3\text{-}4\text{-}\bullet\bullet\bullet\mathrm{g}$ net cumulative claim processes. Claims arrive simultaneously in all 3 dimensions, again according to a Poisson process with rate $\_$. These multivariate claims $\boldsymbol{B}_1\text{-}\boldsymbol{B}_2\text{-}\bullet\bullet\bullet$ are 3-dimensional, componentwise non-negative i.i.d. random vectors, distributed as the generic random vector $\boldsymbol{B}$. We impose the following counterpart of Equation (7.1): in self-evident notation, the following almost-sure ordering applies:

$$\mathrm{P} \quad {}^{1}1^{o} > \quad {}^{1}2^{o} > \quad > \quad {}^{1}3^{o} \quad = 1\bullet \tag{7.12}$$

The premium rate is, for all net cumulative claim processes, equal to A. We define $i\,{}^1\alpha^o$ analogously to (7.2), with $1{}^1\alpha^o$ the 3-dimensional LST of $\boldsymbol{B}$. Just like in the previous section, we focus on the Laplace transform of the joint workload vector. We leave the translation to the Laplace transform of the probability that all firms get ruined before $)_V$ to the reader; cf. (7.6).

B *Derivation of multivariate transform.* The goal of this section is to analyze the transform of the random vector $\boldsymbol{Q}{}^1C^o$, with $\boldsymbol{Q}{}^10^o = \boldsymbol{0}$:

$$\hat{}_C{}^1\boldsymbol{\alpha}^o := \mathrm{E}\,4 \quad {}^{">W^1C^o}\text{-}$$

where, as before, we will settle for its evaluation at the exponentially distributed time $)_V\bullet$ Central objects in our analysis are, with $\boldsymbol{x}_{»8¼} := {}^1G_1\text{-}\bullet\bullet\bullet\text{-}G_8{}^o$ for $8 \ 2 \ \mathrm{f}1\text{-}\bullet\bullet\bullet\text{-}3\mathrm{g}$, the transform

$$\hat{}_C^{\,18^o}\,{}^1\boldsymbol{\alpha}_{»8¼}{}^o := \mathrm{E}\,4 \quad {}^{">W^1C^o}\,1\mathrm{f}\boldsymbol{Q}_{»8¼}{}^1C^o \ ; \ \boldsymbol{0}\text{-}\&_{8\,¸\,1}{}^1C^o = \bullet\bullet\bullet = \&_3{}^1C^o = 0\mathrm{g}$$

$$= \mathbb{E}\, e^{-\boldsymbol{\alpha}_{[i]}^\top \boldsymbol{W}_{[i]}(t)} 1\{\boldsymbol{Q}_{[i]}(t) > \mathbf{0}, Q_{i+1}(t) = \cdots = Q_d(t) = 0\}$$
$$= \mathbb{E}\, e^{-\boldsymbol{\alpha}_{[i]}^\top \boldsymbol{W}_{[i]}(t)} 1\{\boldsymbol{Q}_{[i]}(t) > \mathbf{0}, Q_{i+1}(t) = 0\},$$

where the last equality is a consequence of the ordering $Q_1(t) \geq \cdots \geq Q_d(t)$. Recall that in Section 7.2 we already developed a procedure by which we can identify

$$\xi^{(1)}_{T_q}(\alpha_{[1]}) = \mathbb{E}\, e^{-\alpha_{[1]}^\top Q_{[1]}(T_q)} 1\{Q_{[1]}(T_q) > 0, Q_2(T_q) = 0\}$$
$$= \mathbb{E}\, e^{-\alpha_1 Q_1(T_q)} 1\{Q_1(T_q) > 0, Q_2(T_q) = 0\}.$$

In addition, we know how to compute

$$\xi^{(0)}_{T_q}(\alpha_{[0]}) = p_{T_q} := \mathbb{P}(Q_1(T_q) = \cdots = Q_d(T_q) = 0) = \mathbb{P}(Q_1(T_q) = 0).$$

We essentially follow the approach developed in Section 7.2, so as to find an expression for $\xi_{T_q}(\boldsymbol{\alpha})$; as this procedure is to a large extent analogous, we leave out some details. We thus find, with each all-ones vector $\mathbf{1}$ used in the following expression having the appropriate dimension,

$$\xi_{T_q}(\boldsymbol{\alpha}) = \frac{A \sum_{i=0}^{d-1} \left(\mathbf{1}^\top \boldsymbol{\alpha} - \mathbf{1}^\top \boldsymbol{\alpha}_{[i]}\right) \xi^{(i)}_{T_q}(\boldsymbol{\alpha}_{[i]}) - q}{\varphi(\boldsymbol{\alpha}) - q}. \tag{7.13}$$

The idea is now that we recursively identify the unknown functions in the numerator of (7.13). More concretely, supposing that expressions for

$$\xi^{(0)}_{T_q}(\boldsymbol{\alpha}_{[0]}), \xi^{(1)}_{T_q}(\boldsymbol{\alpha}_{[1]}), \ldots, \xi^{(d-2)}_{T_q}(\boldsymbol{\alpha}_{[d-2]})$$

are available, we point out how to determine $\xi^{(d-1)}_{T_q}(\boldsymbol{\alpha}_{[d-1]})$.

Fixing $\boldsymbol{\alpha}_{[d-1]}$ and q, using the same argumentation as in Section 7.2, we can find a unique α_d (in a certain region) such that $\varphi(\boldsymbol{\alpha}) - q = 0$, which we in the sequel denote by $\psi_d(\boldsymbol{\alpha}_{[d-1]}, q)$. As any such root of the denominator should be a root of the numerator as well, inserting $\alpha_d = \psi_d(\boldsymbol{\alpha}_{[d-1]}, q)$ in the numerator of (7.13) should yield 0; the existence of this root can be proven as in Lemma 7.3. By some algebra we obtain the recursive relation

$$\xi^{(d-1)}_{T_q}(\boldsymbol{\alpha}_{[d-1]}) = \frac{q}{A\,\psi_d(\boldsymbol{\alpha}_{[d-1]}, q)} - \sum_{i=0}^{d-2} \left(\frac{\mathbf{1}^\top \boldsymbol{\alpha}_{[d-1]} - \mathbf{1}^\top \boldsymbol{\alpha}_{[i]}}{\psi_d(\boldsymbol{\alpha}_{[d-1]}, q)} + 1\right) \xi^{(i)}_{T_q}(\boldsymbol{\alpha}_{[i]}).$$

It requires tedious calculations to solve this recursion, eventually yielding the following result. Here we define by $\psi_j(\boldsymbol{\alpha}_{[j-1]}, q)$ the solution for α_j in the equation $\varphi(\boldsymbol{\alpha}_{[j]}, \mathbf{0}) - q = 0$ (with here the vector $\mathbf{0}$ being of dimension $d - j$), for given values of $\boldsymbol{\alpha}_{[j-1]}$ and q.

Theorem 7.2 *For any $\boldsymbol{\alpha} \geq \mathbf{0}$ and $q > 0$,*

$$\bar p_q(\alpha) = \frac{\alpha_1-\psi_1(q)}{\varphi(\alpha)-q}\,\frac{q}{\psi_1(q)}\prod_{j=2}^{d}\frac{\psi_j(\alpha_{[j-1]},q)-\alpha_j}{\psi_j(\alpha_{[j-1]},q)}.$$

▷ *Alternative derivation; quasi-product form.* The multiplicative structure appearing in Theorem 7.2 could have been concluded in an easier way though. To this end, observe that (7.13) entails for the system that consists of just the first $d-1$ net cumulative claim processes, in self-evident notation,

$$\bar p_q(\alpha_{[d-1]},0) = \mathbb{E}\,e^{-\langle\alpha_{[d-1]},W_{[d-1]}(T_q)\rangle} = \frac{r\sum_{i=0}^{d-2}\left(\mathbf{1}^{\top}\alpha_{[d-1]}-\mathbf{1}^{\top}\alpha_{[i]}\right)\bar p_q^{(i)}(\alpha_{[i]})-q}{\varphi(\alpha_{[d-1]},0)-q},$$

where we recall that $\alpha=\alpha_{[d]}$. Combining this equality with Equation (7.13), one directly obtains that

$$\begin{aligned}(\varphi(\alpha_{[d]})-q)\,\bar p_q(\alpha_{[d]}) = r\alpha_d\sum_{i=0}^{d-1}\bar p_q^{(i)}(\alpha_{[i]})\,+\\ (\varphi(\alpha_{[d-1]},0)-q)\,\bar p_q(\alpha_{[d-1]},0).\end{aligned} \tag{7.14}$$

Now plug in $\alpha_{[d]}=(\alpha_{[d-1]},\psi_d(\alpha_{[d-1]},q))$ so that the left-hand side vanishes, and as a consequence the right-hand side as well. This entails that

$$r\,\psi_d(\alpha_{[d-1]},q)\sum_{i=0}^{d-1}\bar p_q^{(i)}(\alpha_{[i]}) = -(\varphi(\alpha_{[d-1]},0)-q)\,\bar p_q(\alpha_{[d-1]},0),$$

and hence

$$\sum_{i=0}^{d-1}\bar p_q^{(i)}(\alpha_{[i]}) = -\frac{(\varphi(\alpha_{[d-1]},0)-q)\,\bar p_q(\alpha_{[d-1]},0)}{r\,\psi_d(\alpha_{[d-1]},q)}. \tag{7.15}$$

Inserting this back into Equation (7.14), and dividing by $\varphi(\alpha_{[d]})-q$, we find the relation

$$\bar p_q(\alpha_{[d]}) = \frac{\varphi(\alpha_{[d-1]},0)-q}{\varphi(\alpha_{[d]})-q}\,\frac{\psi_d(\alpha_{[d-1]},q)-\alpha_d}{\psi_d(\alpha_{[d-1]},q)}\,\bar p_q(\alpha_{[d-1]},0).$$

The next step is to iterate this relation $d-1$ times, and to use that, as a consequence of Theorem 1.1,

$$\bar p_q(\alpha_{[1]},0,\ldots,0) = \mathbb{E}\,e^{-\alpha_1 W_1(T_q)} = \frac{\alpha_1-\psi_1(q)}{\varphi(\alpha_1,0,\ldots,0)-q}\,\frac{q}{\psi_1(q)}.$$

We have thus recovered Theorem 7.2. Using Theorem 7.2 with $d=i$ and (7.15) with $d=i+1$, we also obtain the following relation:

$$\prod_{j=0}^{i} \hat{\Pi}^{(j)}_{\beta}(\alpha[j]) = \frac{U_1 - k_1(\beta)}{A\, l_{i+1}(\alpha[i],\beta)}\,\frac{\beta}{k_1(\beta)} \prod_{j=2}^{i} \frac{l_j(\alpha[j-1],\beta) - U_j}{l_j(\alpha[j-1],\beta)} =: \psi_i(\alpha,\beta).$$

This leads to the following result.

Corollary 7.1 *For any $\alpha[i] > \mathbf{0}$ and $\beta > 0$,*

$$\hat{\Pi}^{(i)}_{\beta}(\alpha[i]) = \psi_i(\alpha,\beta) \,/\, \psi_{i-1}(\alpha,\beta).$$

Given the structure of the result in Theorem 7.2, it is often said that $\boldsymbol{Q}(t)$ is of *quasi-product form*.

7.4 Tandem queueing networks

In this section, we focus on a tandem queueing network fed by a compound Poisson process $J(t)$ with arrival rate $\lambda > 0$. The i.i.d. service requirements $B_1, B_2, \dots$ are distributed as a generic non-negative random variable B with LST $b(\alpha)$; observe that the argument of $b(\alpha)$ is single-dimensional. We consider a network that consists of d queues in series, with (constant) service rates $c_1, \dots, c_d$ that are non-increasing (i.e., $c_1 > c_2 > \cdots > c_d$). The output of the i-th queue is continuously fed into the $(i+1)$-st queue, for $i = 1, \dots, d-1$; no external input arrives at queues $2, \dots, d$. Although this framework is seemingly different from the one discussed in Section 7.3, we argue below that there is a close relationship with the model of Section 7.3, so that the joint workload distribution in the tandem queueing network immediately follows from Theorem 7.2.

▷ *Fitting tandem system in framework of Section 7.3.* In the previous section we have analyzed a multivariate risk system and its dual multivariate queueing system. To fit the above tandem network into this setup, the following observation is crucial. Let $W_i(t)$ be the workload in the i-th queue, with $i = 1, \dots, d$, at time $t > 0$, where we assume that the system starts empty at time 0. Recall that the workload in the first queue obeys

$$W_1(t) = (J(t) - c_1 t) - \inf_{s\in[0,t]} (J(s) - c_1 s).$$

Now consider the *total* workload of the first and second queue, i.e., $W_1(t) + W_2(t)$. It takes some thought to realize that this quantity is only affected by the service rate c_2, and not by c_1. Indeed, c_1 just influences how much of the total workload resides in each of the two queues, whereas c_2 effectively acts as the service rate of the sum of the first and second queue. This means that

$$W_1(t) + W_2(t) = (J(t) - c_2 t) - \inf_{s\in[0,t]} (J(s) - c_2 s).$$

This principle is pictorially illustrated in Figure 7.2. Extending this argument, we obtain for any $i = 1, \dots, d$,

$$Q^{(i)}(t) := \sum_{j=1}^{i} Q_j(t) = (X(t) - r_i t) - \inf_{s\in[0,t]} (X(s) - r_i s).$$

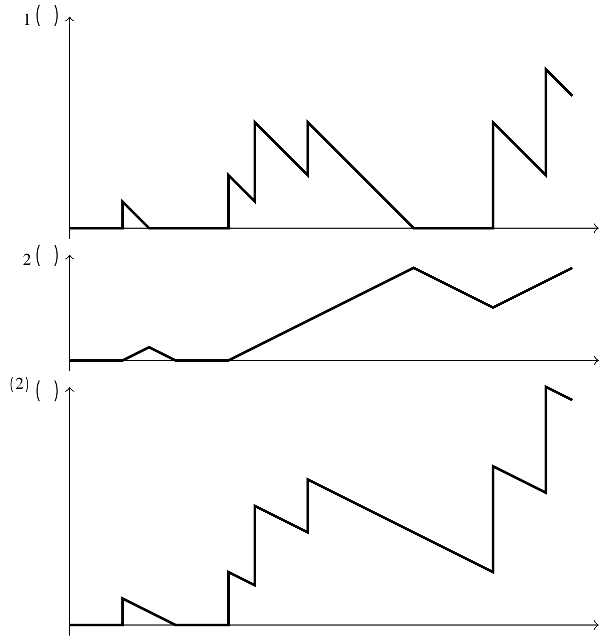

Fig. 7.2 Tandem queueing processes $Q_1(t)$ and $Q_2(t)$, and the sum $Q^{(2)}(t)$. $Q_1(t)$ is an M/G/1 queue with drain rate r_1 and $Q^{(2)}(t)$ is an M/G/1 queue with drain rate r_2. While not empty, $Q_2(t)$ increases at rate $r_1 - r_2$ and decreases at rate r_2.

Now recall Remark 7.1 (and in particular its evident extension to the case that d is larger than 2), and observe that $Q^{(i)}(t)/r_i$ can be seen as the workload in a queue fed by a compound Poisson process with arrival rate λ and i.i.d. service requirements distributed as B/r_i, emptied at a unit rate. But because

$$\mathbb{P}\left(B/r_d > B/r_{d-1} > \cdots > B/r_1\right) = 1,$$

we can apply the theory developed in Section 7.3 to describe the joint distribution of these d workloads (with, compared to Section 7.3, the indices $1,\dots,d$ being swapped), and hence also of the original d workloads $Q_1(t),\dots,Q_d(t)$.

▷ *Derivation of multidimensional transform.* With $\bar{\alpha}_i := \mathbf{1}^\top\alpha_{[d]} - \mathbf{1}^\top\alpha_{[i-1]}$, for $i = 1,\dots,d$, the object of interest is

$$\text{^}_{\mathtt{C}}\,{}^{1}\alpha^{\mathrm{o}} := \mathrm{E}\exp\ \ \tilde{\text{Ö}}_{\mathtt{8}=1}\ U_{\mathtt{8}}\&^{\,1\mathtt{8}^{\mathrm{o}}}\,{}^{1}\mathtt{C}^{\mathrm{o}}\ ! = \mathrm{E}\exp\ \ \tilde{\text{Ö}}_{\mathtt{8}=1}\ \bar{U}_{\mathtt{8}}\&_{\mathtt{8}}\,{}^{1}\mathtt{C}^{\mathrm{o}}\ !\ \bullet$$

As pointed out above, to make sure that the theory of Section 7.3 applies, we work with $\&^{\,1\mathtt{8}^{\mathrm{o}}}\,{}^{1}\mathtt{C}^{\mathrm{o}}\bullet 2_{\mathtt{8}}$ rather than $\&^{\,1\mathtt{8}^{\mathrm{o}}}\,{}^{1}\mathtt{C}^{\mathrm{o}}$. To this end, we rewrite

$$\text{^}_{\mathtt{C}}\,{}^{1}\alpha^{\mathrm{o}} = \mathrm{E}\exp\ \ \tilde{\text{Ö}}_{\mathtt{8}=1}\ U_{\mathtt{8}}2_{\mathtt{8}}\frac{\&^{\,1\mathtt{8}^{\mathrm{o}}}\,{}^{1}\mathtt{C}^{\mathrm{o}}}{2_{\mathtt{8}}}\ !\ \bullet$$

We define by $i\,{}^{1}\alpha^{\mathrm{o}}$ the Laplace exponent of the net input process of the 3-dimensional queueing process ${}^{1}\&^{\,11^{\mathrm{o}}}\,{}^{1}\mathtt{C}^{\mathrm{o}}\bullet 2_1 \text{-} \bullet\bullet\bullet \text{-} \&^{\,13^{\mathrm{o}}}\,{}^{1}\mathtt{C}^{\mathrm{o}}\bullet 2_3{}^{\mathrm{o}}$:

$$i\,{}^{1}\alpha^{\mathrm{o}} := \mathbf{1}^{>}\alpha\quad \_\,{}^{1}1\quad 1\,{}^{1}\mathbf{1}^{>}\alpha\ ^{\mathrm{oo}}\text{-}$$

where α is the 3-dimensional vector whose $\mathtt{8}$-th entry is $U_{\mathtt{8}}\bullet 2_{\mathtt{8}}$.

As noted above, we have that $\&^{\,13^{\mathrm{o}}}\,{}^{1}\mathtt{C}^{\mathrm{o}}\bullet 2_3 > \quad > \&^{\,11^{\mathrm{o}}}\,{}^{1}\mathtt{C}^{\mathrm{o}}\bullet 2_1$, so that when applying Theorem 7.2 we have to swap the indices. We let $k_3\,{}^{1}V^{\mathrm{o}}$ be the inverse of $i\,{}^{1}\mathbf{0}\text{-}\,U_3{}^{\mathrm{o}}$. Also, $I_{\mathtt{9}}\,{}^{1}U_{\mathtt{9}\,\text{¸}\,1}\text{-}\bullet\bullet\bullet\text{-}U_3\text{-}\,V^{\mathrm{o}}$ is the unique solution for $U_{\mathtt{9}}$ in $i\,{}^{1}\mathbf{0}\text{-}\,U_{\mathtt{9}}\text{-}\bullet\bullet\bullet\text{-}U_3{}^{\mathrm{o}}\quad V = 0$, for given $U_{\mathtt{9}\,\text{¸}\,1}\text{-}\bullet\bullet\bullet\text{-}U_3$ in a specific region and V. We find the following result that provides an expression for $\text{^}_{\mathtt{C}}\,{}^{1}\alpha^{\mathrm{o}}$ at the exponentially distributed time $)_V$, again revealing a quasi-product form structure. Here $\alpha_{\text{›}}$ is the 3-dimensional vector whose $\mathtt{8}$-th entry is $U_{\mathtt{8}}2_{\mathtt{8}}$.

Theorem 7.3 *For any $\alpha > \mathbf{0}$ and V ¡ 0,*

$$\text{^}_{)_V}\,{}^{1}\alpha^{\mathrm{o}} = \frac{U_{\grave{3}}\quad k_3\,{}^{1}V^{\mathrm{o}}}{i\,{}^{1}\alpha_{\text{›}}{}^{\mathrm{o}}\quad V}\ \frac{V}{k_3\,{}^{1}V^{\mathrm{o}}}\ \prod_{\mathtt{9}=1}^{3\text{Ö}1}\frac{I_{\mathtt{9}}\,{}^{1}U_{\grave{\mathtt{9}}\,\text{¸}\,1}\text{-}\bullet\bullet\bullet\text{-}U_{\grave{3}}\text{-}\,V^{\mathrm{o}}\quad U_{\grave{\mathtt{9}}}}{I_{\mathtt{9}}\,{}^{1}U_{\grave{\mathtt{9}}\,\text{¸}\,1}\text{-}\bullet\bullet\bullet\text{-}U_{\grave{3}}\text{-}\,V^{\mathrm{o}}}\bullet$$

7.5 Multivariate Gerber-Shiu metrics

Whereas in previous sections we have primarily focused on the joint ruin probability, in this section we aim at obtaining more detailed insight into the ruin behavior. This we do by studying the joint distribution of ruin times of both insurance firms (so taking $3 = 2$), together with the corresponding undershoots and overshoots, i.e., the so-called (multivariate) *Gerber-Shiu metrics.* As it turns out, the corresponding joint transforms can be evaluated in closed form, albeit at the expense of performing a rather intricate analysis. In this section, we consider the bivariate case, but the analysis can be extended to higher dimensions, as long as the claim sizes are almost surely ordered (i.e., fulfil relation (7.12)).

B *Notation.* We abbreviate $\boldsymbol{u} = {}^{1}D_1\text{-}\,D_2{}^{\mathrm{o}>}\ 2\ \text{»}0\text{-}\ 1\ {}^{\mathrm{o}2}$, $\boldsymbol{Y}\,{}^{1}\mathtt{C}^{\mathrm{o}} = {}^{1}\,.\,{}_1{}^{1}\mathtt{C}^{\mathrm{o}}\text{-}\,.\,{}_2{}^{1}\mathtt{C}^{\mathrm{oo}>}$, and

$$\boldsymbol{\tau}\,{}^{1}\boldsymbol{u}^{\mathrm{o}} := \begin{matrix} g_1\,{}^{1}D_1{}^{\mathrm{o}} \\ g_2\,{}^{1}D_2{}^{\mathrm{o}} \end{matrix}\ \text{-}\quad \boldsymbol{Y}\,{}^{1}\boldsymbol{\tau}\,{}^{1}\boldsymbol{u}^{\mathrm{o}}\ {}^{\mathrm{o}} := \begin{matrix} .\,{}_1{}^{1}g_1\,{}^{1}D_1{}^{\mathrm{o}}\ {}^{\mathrm{o}} \\ .\,{}_2{}^{1}g_2\,{}^{1}D_2{}^{\mathrm{o}}\ {}^{\mathrm{o}} \end{matrix}\ \text{-}\quad \boldsymbol{Y}\,{}^{1}\boldsymbol{\tau}\,{}^{1}\boldsymbol{u}^{\mathrm{oo}} := \begin{matrix} .\,{}_1{}^{1}g_1\,{}^{1}D_1{}^{\mathrm{oo}} \\ .\,{}_2{}^{1}g_2\,{}^{1}D_2{}^{\mathrm{oo}} \end{matrix}\ ;$$

here $\tau_i(u_i)$ is the ruin time corresponding to the net cumulative claim process $Y_i(t)$, i.e., the smallest $t > 0$ such that $Y_i(t) > u_i$, for $i = 1, 2$.

In this section the primary object of study is, for $\boldsymbol{u} \geqslant 0$ and $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \geqslant \mathbf{0}$, $\boldsymbol{\gamma}_3 \leqslant \mathbf{0}$ (where $\boldsymbol{\gamma}_i = (\gamma_{i1}, \gamma_{i2})^\top$ for $i = 1, 2, 3$),

$$\begin{aligned}\xi(\boldsymbol{u}) &\equiv \xi(\boldsymbol{u}, q, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3)\\ &:= \mathbb{E}\, e^{-\boldsymbol{\gamma}_1^\top \boldsymbol{\tau}(\boldsymbol{u}) - \boldsymbol{\gamma}_2^\top (\boldsymbol{u} - \boldsymbol{Y}(\boldsymbol{\tau}(\boldsymbol{u})-)) - \boldsymbol{\gamma}_3^\top (\boldsymbol{u} - \boldsymbol{Y}(\boldsymbol{\tau}(\boldsymbol{u})))}\, 1\{\boldsymbol{\tau}(\boldsymbol{u}) \leqslant e_q \mathbf{1}\}.\end{aligned}$$

In the sequel we frequently suppress the parameters $q, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3$, as these are held constant throughout the analysis. Observe that $\boldsymbol{u} - \boldsymbol{Y}(\boldsymbol{\tau}(\boldsymbol{u})-)$ is inherently non-negative, and can be interpreted as the vector of the undershoots; also, $\boldsymbol{u} - \boldsymbol{Y}(\boldsymbol{\tau}(\boldsymbol{u}))$ is inherently non-positive, and can be seen as the vector of (the negatives of) the overshoots. By $\{\boldsymbol{\tau}(\boldsymbol{u}) \leqslant e_q \mathbf{1}\}$ we mean that both ruin times (i.e., $\tau_1(u_1)$ and $\tau_2(u_2)$) should be smaller than the independently sampled exponentially distributed random variable e_q.

We analyze $\xi(\boldsymbol{u})$ through its (nine-fold) transform

$$c(\boldsymbol{\alpha}) \equiv c(\boldsymbol{\alpha}, q, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3) := \int_0^\infty \int_0^\infty e^{-\boldsymbol{\alpha}^\top \boldsymbol{u}}\, \xi(\boldsymbol{u}, q, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3)\, \mathrm{d}u_1\, \mathrm{d}u_2,$$

where $\boldsymbol{\alpha} \geqslant \mathbf{0}$ and $q > 0$.

∘ *Derivation of integro-differential equation.* Denote by

$$\begin{aligned}\xi_i(u) &\equiv \xi(u, q, \gamma_{1i}, \gamma_{2i}, \gamma_{3i})\\ &:= \mathbb{E}\, e^{-\gamma_{1i}\tau_i(u) - \gamma_{2i}(u - Y_i(\tau_i(u)-)) - \gamma_{3i}(u - Y_i(\tau_i(u)))}\, 1\{\tau_i(u) \leqslant e_q\},\end{aligned}$$

for $i = 1, 2$, the univariate counterparts of $\xi(\boldsymbol{u})$; these describe the ruin times, undershoots and overshoots of the individual firms. Recall that we know the transform (with respect to u) of these objects $\xi_i(u)$, which in the sequel we denote by $c_i(\alpha)$; see Exercise 1.2. To keep our computations somewhat compact, in the sequel we use the shorthand notation

$$\mathbb{P}(\boldsymbol{B} \in \mathrm{d}\boldsymbol{v}) := \mathbb{P}(B^{(1)} \in \mathrm{d}v_1, B^{(2)} \in \mathrm{d}v_2).$$

The main equation that our analysis is based on relates $\xi(\boldsymbol{u})$ to the same object, but then evaluated in different arguments. Concretely, as $\Delta t \downarrow 0$, by considering the scenarios of a claim arrival and killing, we obtain

$$\begin{aligned}\xi(\boldsymbol{u}) = e^{-\boldsymbol{\gamma}_1^\top \mathbf{1}\,\Delta t} \Big(&\lambda\,\Delta t \int_{v_1=0}^{u_1}\int_{v_2=0}^{u_2} \xi(\boldsymbol{u} - \boldsymbol{v})\, \mathbb{P}(\boldsymbol{B} \in \mathrm{d}\boldsymbol{v})\\ &+ \lambda\,\Delta t \int_{v_1=0}^{u_1}\int_{v_2=u_2}^{\infty} \xi_1(u_1 - v_1)\, e^{-\gamma_{22}u_2 - \gamma_{32}(u_2 - v_2)}\, \mathbb{P}(\boldsymbol{B} \in \mathrm{d}\boldsymbol{v})\\ &+ \lambda\,\Delta t \int_{v_1=u_1}^{\infty}\int_{v_2=0}^{u_2} \xi_2(u_2 - v_2)\, e^{-\gamma_{21}u_1 - \gamma_{31}(u_1 - v_1)}\, \mathbb{P}(\boldsymbol{B} \in \mathrm{d}\boldsymbol{v})\,,\end{aligned}$$

$$-\lambda\Delta t\int_{v_1=u_1}^{\infty}\int_{v_2=u_2}^{\infty} e^{-\beta_2^{\top}u-\beta_3^{\top}(u-v)}\,\mathbb{P}(B\in \mathrm{d}v)+$$
$$(1-(\lambda+\delta)\Delta t)\,\phi(u+r\mathbf{1}\Delta t)+o(\Delta t). \tag{7.16}$$

for $u>\mathbf{0}$; cf. the procedure followed in Exercise 1.2. (Note that in the four double integrals appearing in (7.16) we could have restricted ourselves to $v_1>v_2$, but for notational convenience we leave this out; this means that the bivariate density $\mathbb{P}(B\in\mathrm{d}v)$ is assumed 0 for $v_1\leqslant v_2$.)

From this point on, we follow the standard procedure. By subtracting $\phi(u+r\mathbf{1}\Delta t)$ from both sides, dividing by Δt, letting $\Delta t\downarrow 0$, and performing some algebraic manipulations, we readily arrive at the following integro-differential equation: for $u>\mathbf{0}$,

$$r\frac{\partial}{\partial u_1}\phi(u)+\frac{\partial}{\partial u_2}\phi(u)=-\lambda\int_0^{u_1}\int_0^{u_2}\phi(u-v)\,\mathbb{P}(B\in\mathrm{d}v)$$
$$-\lambda\int_{u_1}^{\infty}\int_0^{u_2}\phi_1(u_1-v_1)\,e^{-\beta_{22}u_2-\beta_{32}(u_2-v_2)}\,\mathbb{P}(B\in\mathrm{d}v)$$
$$-\lambda\int_0^{u_1}\int_{u_2}^{\infty}\phi_2(u_2-v_2)\,e^{-\beta_{21}u_1-\beta_{31}(u_1-v_1)}\,\mathbb{P}(B\in\mathrm{d}v)$$
$$-\lambda\int_{u_1}^{\infty}\int_{u_2}^{\infty}e^{-\beta_2^{\top}u-\beta_3^{\top}(u-v)}\,\mathbb{P}(B\in\mathrm{d}v)-(\mathbf{1}^{\top}\gamma_1+\lambda+\delta)\,\phi(u).$$

This is the bivariate counterpart of what was found in part (ii) of Exercise 1.2. Note that we have not yet used the ordering assumption (7.12).

▷ *Identification of double transform.* With the above integro-differential equation at our disposal, we proceed by performing the (double) transform with respect to u; cf. parts (iii) and (iv) of Exercise 1.2. To this end, we multiply the full equation by $e^{-\alpha^{\top}u}$ (with $\alpha>\mathbf{0}$) and integrate over the non-negative u_1 and u_2. It requires some standard calculus to verify that the sum of the five terms on the right-hand side can be rewritten as

$$(\lambda b(\alpha)-\mathbf{1}^{\top}\gamma_1-\lambda-\delta)\,c(\alpha)+\lambda Z(\alpha),$$

where

$$Z(\alpha):=$$
$$c_1(\alpha_1)\frac{b(\alpha_1,-\beta_{32})-b(\alpha_1,\alpha_2+\beta_{22})}{\alpha_2+\beta_{22}+\beta_{32}}+c_2(\alpha_2)\frac{b(-\beta_{31},\alpha_2)-b(\alpha_1+\beta_{21},\alpha_2)}{\alpha_1+\beta_{21}+\beta_{31}}+$$
$$\frac{b(-\beta_{31},-\beta_{32})-b(-\beta_{31},\alpha_2+\beta_{22})-b(\alpha_1+\beta_{21},-\beta_{32})+b(\alpha_1+\beta_{21},\alpha_2+\beta_{22})}{(\alpha_1+\beta_{21}+\beta_{31})(\alpha_2+\beta_{22}+\beta_{32})}.$$

The left-hand side can be evaluated relying on integration by parts. A direct computation yields

$$r\mathbf{1}^{\top}\alpha\,c(\alpha)+rc_1(\alpha_2)+rc_2(\alpha_1).$$

where

$$c_1(\alpha) := \int_0^\infty p(0,u)\, e^{-\alpha u}\, du, \qquad c_2(\alpha) := \int_0^\infty p(u,0)\, e^{-\alpha u}\, du.$$

Upon combining the above, recalling that $\varphi(\boldsymbol\alpha) = r\,\mathbf{1}^\top\boldsymbol\alpha - \lambda(1 - b(\boldsymbol\alpha))$, we find the following result.

Proposition 7.1 *For any* $\boldsymbol\alpha > \mathbf{0}$, $\beta \geq 0$, $\boldsymbol\gamma_1, \boldsymbol\gamma_2 > \mathbf{0}$, $\boldsymbol\gamma_3 \leq \mathbf{0}$,

$$c(\boldsymbol\alpha) = \frac{r(c_1(\alpha_2) + c_2(\alpha_1)) - \zeta(\boldsymbol\alpha)}{\varphi(\boldsymbol\alpha) - \mathbf{1}^\top\boldsymbol\gamma_1 - \beta}. \tag{7.17}$$

We have, however, not yet identified the functions $c_i(\alpha)$. The key idea is that the almost sure ordering (7.12), which effectively entails that $X_1(t) \geq X_2(t)$, can be used to evaluate $c_1(\alpha)$, where a crucial role is played by the (easily verified) fact that $\tau_1(0) \leq \tau_2(u)$ for all $u \geq 0$. Then, by Lemma 7.3, we obtain that

$$c_2(\alpha) = c_1\left(\psi_2(\alpha, \mathbf{1}^\top\boldsymbol\gamma_1 + \beta) + \frac{1}{r}\,\zeta\left(\alpha, \psi_2(\alpha, \mathbf{1}^\top\boldsymbol\gamma_1 + \beta)\right)\right). \tag{7.18}$$

▷ *Derivation of auxiliary transform.* We proceed by pointing out how the auxiliary transform $c_1(\alpha)$ can be evaluated. Throughout we will extensively work with the random vector

$$\boldsymbol Z(u) \equiv (Z_1(u), Z_2(u))^\top := (X_2(\tau_1(u)), \Delta_2(u))^\top,$$

where $\Delta_2(u)$ is the size of the claim in the net cumulative claim process $X_2(t)$ at the ruin time $\tau_1(u)$ of the net cumulative claim process $X_1(t)$. In our analysis, an important role is played by the transform of the ruin time, undershoot and overshoot related to the process $X_1(t)$, jointly with this random vector $\boldsymbol Z(u)$:

$$p_1(u, d\boldsymbol z) := \mathbb{E}\, e^{-\mathbf{1}^\top\boldsymbol\gamma_1\tau_1(u) - \gamma_{21}(u - X_1(\tau_1(u))) - \gamma_{31}(u - X_1(\tau_1(u)-))}\, I(u, d\boldsymbol z),$$

with $I(u, d\boldsymbol z) := 1\{\tau_1(u) \leq T_\beta, \boldsymbol Z(u) \in d\boldsymbol z\}$.

The key identity that we will be working with is

$$\begin{aligned} p(0,u) = &\int_{z_1=u}^{\infty}\int_{z_2=0}^{\infty} p_1(0, d\boldsymbol z)\, e^{-\gamma_{22}(u - z_1 + z_2) - \gamma_{32}(u - z_1)} \\ &+ \int_{z_1=-\infty}^{u}\int_{z_2=0}^{\infty} p_1(0, d\boldsymbol z)\, p_2(u - z_1); \end{aligned} \tag{7.19}$$

here the first term on the right-hand side corresponds to the scenario that $X_2(t)$ exceeds u for the first time at $\tau_1(0)$ (i.e., $\tau_1(0) = \tau_2(u)$), and the second term to the scenario that u is not yet exceeded by $X_2(t)$ at time $\tau_1(0)$ (i.e., $\tau_1(0) < \tau_2(u)$); here we use the property that $\tau_1(0) \leq \tau_2(u)$ for all $u \geq 0$, so that $\tau_1(0) > \tau_2(u)$ cannot occur. See Figure 7.3 for an illustration of both possible scenarios. To understand the decomposition (7.19), the following observations are useful. In the first place, in case $g_1(0) = g_2(D)$, the undershoot is

$$
\begin{aligned}
D \quad ._2(g_2(D) \quad) &= D \quad ._2(g_1(0) \quad) \\
&= D \quad ._2(g_1(0)) \, , \quad _2(0) = D \quad /_1(0) \, , \, /_2(0),
\end{aligned}
$$

and the corresponding overshoot is

$$D \quad ._2(g_2(D)) = D \quad ._2(g_1(0)) = D \quad /_1(0) .$$

Secondly, in case $g_1(0) \ddot{Y} g_2(D)$, at $g_1(0)$ the second component still needs to bridge a residual distance of $D \quad ._2(g_1(0)) = D \quad /_1(0)$.

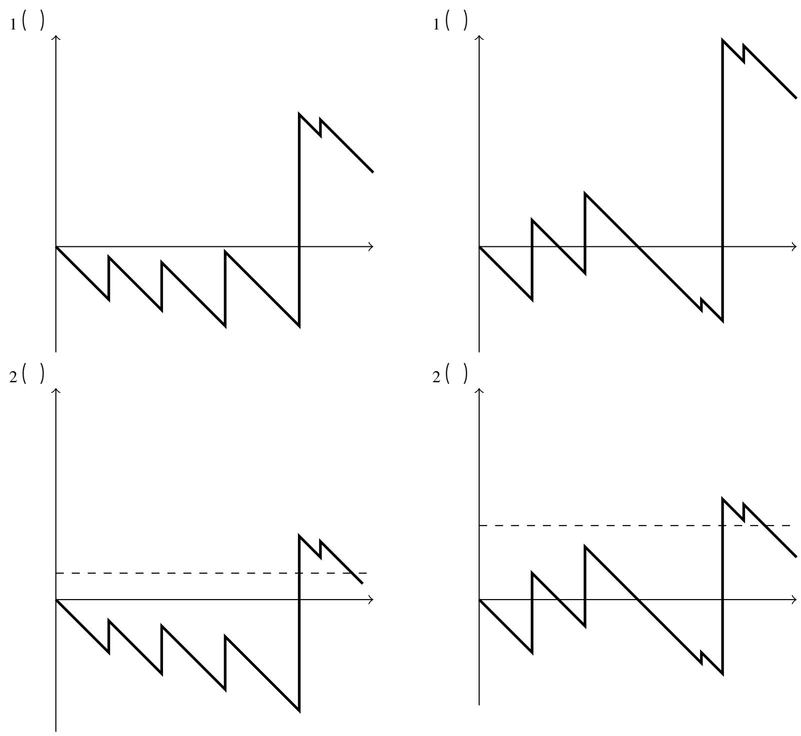

Fig. 7.3 Net cumulative claim processes $._1(C)$ and $._2(C)$ such that $._1(C) > ._2(C)$ for all $C > 0$. The left panels display a scenario of $(._1(C), ._2(C))$ in which $g_1(0) = g_2(D)$, whereas the right panels display a scenario of $(._1(C), ._2(C))$ in which $g_1(0) \ddot{Y} g_2(D)$.

By appealing to (7.19), we thus find that the object of our interest equals

$$c_1(U) = \int_0^{1} 4^{\,UD} \sum_{l_1=D}^{1} \sum_{l_2=0}^{1} ?_1(0, \mathrm{d}z) \, 4^{\,W_{22}(D \quad l_1, l_2) \quad W_{32}(D \quad l_1)} \, \mathrm{d}D \, ,$$

$$\int_0^\infty e^{-\alpha u}\int_{y_1=-\infty}^{u}\int_{y_2=0}^{\infty} p_1(0,\mathrm{d}z)\, p_2(u-y_1)\,\mathrm{d}u. \tag{7.20}$$

Define, for $\boldsymbol{\beta} = (\beta_1,\beta_2)^\top \in \mathbb{R}^2$, the transform

$$\begin{aligned} b(\boldsymbol{\beta}) &:= \mathbb{E}\, e^{-\vartheta_1\tau_1(0) - W_{21}(\ell_1(\tau_1(0))) - W_{31}(\ell_1(\tau(0))) - \beta_1\ell_2(\tau_1(0)) - \beta_2(\tau_1(0))}\,1\{\tau_1(0)<\infty\} \\ &= \mathbb{E}\, e^{-\vartheta_1\tau_1(0) - W_{21}(\ell_1(\tau_1(0))) - W_{31}(\ell_1(\tau(0))) - \boldsymbol{\beta}^\top \ell(0)}\,1\{\tau_1(0)<\infty\} \\ &= \int_{y_1=-\infty}^{\infty}\int_{y_2=0}^{\infty} p_1(0,\mathrm{d}z)\, e^{-\boldsymbol{\beta}^\top z}. \end{aligned}$$

Conclude that if we are able to compute $b(\boldsymbol{\beta})$ for $\boldsymbol{\beta} > \mathbf{0}$, we have (albeit implicitly) identified $p_1(0,\mathrm{d}z)$, enabling us to compute $\bar{c}_1(\alpha)$; recall that we know $p_2(u-y_1)$, appearing in (7.20) (via its transform, that is).

In order to evaluate the quantity $b(\boldsymbol{\beta})$, we proceed by first studying a related object, from which later $b(\boldsymbol{\beta})$ can be found. To this end, we define

$$\begin{aligned} p_1(u) &\equiv p_1(u,\boldsymbol{\beta}) \\ &:= \mathbb{E}\, e^{-\vartheta_1\tau_1(u) - W_{21}(u-\ell_1(\tau_1(u))) - W_{31}(u-\ell_1(\tau(u))) - \boldsymbol{\beta}^\top \ell(u)}\,1\{\tau_1(u)<\infty\}. \end{aligned}$$

Due to the evident identity $b(\boldsymbol{\beta}) = p_1(0,\boldsymbol{\beta})$, we observe that it suffices to have access to $p_1(0,\boldsymbol{\beta})$.

The required quantity $p_1(0,\boldsymbol{\beta})$ can be identified as follows. Relying on the precise same procedure as in Exercise 1.2, we can determine the transform of $p_1(u)$. As a first step, observe that, as $\Delta t\downarrow 0$,

$$\begin{aligned} p_1(u) = e^{-\vartheta_1\Delta t + r_1\Delta t}\Big(&\lambda\Delta t\int_{v_1=0}^{u}\int_{v_2=0}^{\infty}\mathbb{P}(\boldsymbol{B}\in\mathrm{d}\boldsymbol{v})\, p_1(u-v_1)\, e^{-\beta_2 v_2} \\ &+ \lambda\Delta t\int_{v_1=u}^{\infty}\int_{v_2=0}^{\infty}\mathbb{P}(\boldsymbol{B}\in\mathrm{d}\boldsymbol{v})\, e^{-W_{21}u}\, e^{-W_{31}(u-v_1)}\, e^{-\boldsymbol{\beta}^\top \boldsymbol{v}} \\ &+ (1-\lambda\Delta t)\, p_1(u+r_1\Delta t)\Big), \end{aligned} \tag{7.21}$$

up to $o(\Delta t)$-terms. Here, the first double integral corresponds to the scenario that there is a claim arrival but that, despite this claim, $\ell_1(t)$ remains below u, whereas the second double integral corresponds to the scenario that there is a claim arrival by which $\ell_1(t)$ exceeds u; see also Exercise 7.6. Then, as usual, we subtract $p_1(u+r_1\Delta t)$ from both sides of (7.21), divide by Δt, and let $\Delta t\downarrow 0$, so as to obtain an integro-differential equation. By multiplying this equation by $e^{-\alpha u}$ and integrating over u, we obtain after considerable calculus that

$$\bar{\pi}_1(\alpha) := \int_0^\infty e^{-\alpha u}\, p_1(u)\,\mathrm{d}u$$

equals

$$\frac{1}{i\,{}^1U\text{-}X_1{}^{\text{o}}\quad \mathbf{1}^{>}\gamma_1\quad V}\; A\; ?_1{}^10^{\text{o}}\quad -\frac{1^1\;\; W_{31}\text{-}\mathbf{1}^{>}\delta^{\text{o}}\quad 1^1U\;¸\;W_{21}\text{-}\mathbf{1}^{>}\delta^{\text{o}}}{U\;¸\;W_{21}\;¸\;W_{31}}\;\bullet \tag{7.22}$$

Expression (7.22) can be used to determine $?_1{}^10^{\text{o}}\quad ?_1{}^10\text{-}\delta^{\text{o}}$, as follows. Recall the principle that for any value of U (with non-negative real part, that is) for which $i\,{}^1U\text{-}X_1{}^{\text{o}}\quad \mathbf{1}^{>}\gamma_1\quad V$ equals zero, the term between brackets in (7.22) should be equal to zero as well. Using the compact notation $U\quad U\;{}^1V\text{-}\gamma_1\text{-}X_1{}^{\text{o}} := k_1{}^1\mathbf{1}^{>}\gamma_1\;¸\;V^{\text{o}}$, with $V\;7!\;\;k_1{}^1V^{\text{o}}$ denoting the right-inverse of $U\;7!\;\;i\,{}^1U\text{-}X_1{}^{\text{o}}$, this implies that

$$?_1{}^10\text{-}\delta^{\text{o}} = b^1\delta^{\text{o}} = \frac{=}{A}\,\frac{1^1\;\; W_{31}\text{-}\mathbf{1}^{>}\delta^{\text{o}}\quad 1^1U\;\;¸\;W_{21}\text{-}\mathbf{1}^{>}\delta^{\text{o}}}{U\;\;¸\;W_{21}\;¸\;W_{31}}\bullet \tag{7.23}$$

The conclusion is that we have found all ingredients that allow the evaluation of $c_1{}^1U^{\text{o}}$.

Proposition 7.2 *For any* $U > 0$, $V\;¡\;0$, $\gamma_1\text{-}\gamma_2 > \mathbf{0}$, $\gamma_3\;6\;\mathbf{0}$, *we have that* $c_1{}^1U^{\text{o}}$ *is given by Equation* (7.20), *where the transform* $b^1\delta^{\text{o}}$ *is given by Equation* (7.23).

B *Main result.* We have thus arrived at the main result, describing the Gerber-Shiu metrics in the coupled risk system. It uniquely characterizes the joint distribution of the ruin times, undershoots and overshoots pertaining to the two net cumulative claim processes $._1{}^1C^{\text{o}}$ and $._2{}^1C^{\text{o}}$. The idea is that (in principle) the transform $b^1\delta^{\text{o}}$ uniquely defines $?_1{}^10\text{-}\,\mathrm{d}z^{\text{o}}$, through which we can evaluate $c_1{}^1U^{\text{o}}$ via Equation (7.20), which enables the calculation of transform $c_2{}^1U^{\text{o}}$ via Equation (7.18), after which $c^1\alpha^{\text{o}}$ follows from (7.17).

Theorem 7.4 *For any* $\alpha > \mathbf{0}$, $V\;¡\;0$, $\gamma_1\text{-}\gamma_2 > \mathbf{0}$, $\gamma_3\;6\;\mathbf{0}$, *we have that* $c^1\alpha^{\text{o}}$ *is given by* (7.17). *The transform* $c_1{}^1U^{\text{o}}$ *follows from Proposition* 7•2, *and the transform* $c_2{}^1U^{\text{o}}$ *from Equation* (7.18).

7.6 Discussion and bibliographical notes

The results covered by this chapter to a large extent align with those presented in [6], where it is noted that [6] only considers the infinite-horizon case (i.e., no exponential killing). Specifically, the duality result of Lemma 7.1 is taken from [6]. For analyses of the two-dimensional case, in the spirit of the one presented in Section 7.2, we refer to e.g. [2, 3, 10]. For papers on so-called stochastic fluid networks, covering the results of Section 7.4, see a paper by Kella and Whitt [17] and a sequence of papers by Kella [13, 14, 15, 16], as well as the more extensive analysis of [9]. It is noted that the latter stream of papers considers more general Lévy-driven models (where all Lévy processes involved are spectrally one-sided). The approach and results of Section 7.5 are novel.

A crucial role in all papers mentioned above is played by an ordering of the processes involved, similar to the almost sure ordering we have imposed on our processes . ₁¹C⁰-•••-. ₃¹C⁰. The most notable example of an instance in which the ordering naturally arises is that of *reinsurance* [2, 3, 5].

Without imposing the ordering condition, the analysis is considerably more complicated. For an account of results concerning this more general case, we refer to [6, Section 6]; in particular, the approaches developed (in increasing generality) in [4, 8, 7] are there being discussed. In the recent contribution [1] for the bivariate case, an approached based on Laguerre expansions has been followed. It is noted that results on tandem queues can be translated in a straightforward manner into results on related priority queues; see e.g. the explanation of this property in [18]. A recursive procedure for determining Gerber-Shiu metrics in the bivariate context can be found in e.g. [12], where it is not assumed that the ordering condition is in place.

Exercises

7.1 Prove Equation (7.6).

7.2 Verify that $U_2 = 0$ (respectively $U_1 = 0$) in Theorem 7.1 yields the Laplace transform of the ruin probability for insurance company 1 (respectively 2). In particular, observe that (with an obvious notation) $I_2$¹0- V⁰ = $k_2$¹V⁰.

7.3 Verify the statement in Remark 7.2 that * ¹$U_1$⁰ = 1 ¹U_1 ¸ $I_2$¹U_1- 0⁰⁰•_, as well as the decomposition (7.11).

7.4 Verify Equation (7.13).

7.5 Consider the case that ¹1⁰ is exponentially distributed with parameter `, and ¹2⁰ 2 ¹1⁰ for 0 Ÿ 2 6 1. Determine $I_2$¹U_1- V⁰ and ^)$_V$¹α⁰.

7.6 ¹¢⁰ Verify Equation (7.21).
(*Hint*: note that the components of W¹D⁰ can be assigned their values when . ₁¹C⁰ exceeds D, which can only happen due to a claim such that the size of the first component is at least D.)

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Chapter 8
Arrival processes with clustering

Abstract — In this chapter we focus on the Cramér-Lundberg model driven by a claim arrival process with randomly fluctuating rate. We consecutively discuss models in which the arrival rate evolves as an M/G/1 queue (to do justice to a fluctuating number of customers), as a shot-noise process (to model the impact of catastrophic events) and as a Hawkes process (to model the effect of claims triggering additional claims). The main objective is to determine, in the light-tailed context, the decay rate of the ruin probability. The proofs rely either on applying a change-of-measure, or on a large deviations based argumentation.

8.1 Introduction

In this chapter we consider a variant of the Cramér-Lundberg model in which the Poisson claim arrival process is replaced by an arrival process that exhibits some 'random clustering'. We consecutively discuss three specific, relevant arrival processes. In all of them, the Poisson arrival rate is itself a stochastic process; such an arrival process is often referred to as a *Cox process*.

In the first place we focus on a model where the number of customers is fluctuating over time according to a so-called M/G/1 queue (a queue with Poisson arrivals, i.i.d. service times with a general distribution and infinitely many servers; see Appendix B.2 for a brief exposition of this queue).

Then we let the Poisson arrival rate be described by a shot-noise process.

Finally we consider the case in which the Poisson arrival rate follows a Hawkes process, a process that induces what is called 'self excitation'.

An exact analysis of $?^{1}D^{\circ}$ or $?^{1}D\text{-})_{V}^{\circ}$ seems prohibitively difficult. Hence, in this chapter, we focus on deriving — quite powerful — asymptotic results regarding the decay rate of the ruin probability. In our analysis, an important role is played by what could be referred to as the 'limiting Laplace exponent': with $.\,^{1}C^{\circ}$, as before, denoting the net cumulative claim process,

$$\Lambda(\vartheta) := \lim_{t\to\infty}\frac{1}{t}\log \mathbb{E}\, e^{\vartheta X(t)}.$$

We throughout assume that the net-profit condition (see Remark 1.1) holds, which means that

$$\lim_{t\to\infty}\frac{\mathbb{E}\, X(t)}{t} = \Lambda'(0) < 0.$$

In addition, we will work with the corresponding Legendre transform: for $a > 0$,

$$I(a) := \sup_{\vartheta\in\mathbb{R}} \big(\vartheta a - \Lambda(\vartheta)\big). \tag{8.1}$$

This function can be proven to be non-negative and convex, and attains its minimal value 0 at $a = \Lambda'(0)$. In addition, it can be shown that for $a > \Lambda'(0)$, we can restrict ourselves to optimizing over $\vartheta > 0$. These properties are established in Exercise 8.1.

For all three arrival processes under consideration, we shall prove that

$$\lim_{u\to\infty}\frac{1}{u}\log p(u) = -\vartheta^\star,$$

where $\vartheta^\star > 0$ is such that $\Lambda(\vartheta^\star) = 0$. This will be done by first observing that $-\vartheta^\star$ is a lower bound on the logarithmic decay rate of $p(u)$ — a result that follows rather easily — and then proving the considerably harder result that $-\vartheta^\star$ is an upper bound.

Remark 8.1 The fact that $-\vartheta^\star$ is a lower bound on the decay rate of $p(u)$ can be derived employing essentially the same reasoning as the one used in the large deviations based approach at the end of Section 2.2; importantly, as the increments are now not i.i.d., instead of Cramér's theorem the Gärtner-Ellis theorem [6, Section 2.3] needs to be used. Formally, application of the Gärtner-Ellis theorem requires the verification of a specific technical condition, the so-called *steepness condition*; in this chapter we assume that this condition is met (but see Exercise 8.5 for a verification for a special instance of the model with Hawkes driven arrivals). Then, as in Section 2.2, it can be argued that

$$\inf_{\tau>0}\tau\, I(1/\tau) = \vartheta^\star.$$

With $\tau^\star$ denoting the optimizing τ, $\alpha^\star := 1/\tau^\star$ can be interpreted as the 'cheapest' slope to reach a high level, and $\tau^\star u$ is a proxy for the typical time it takes to exceed level u. It is substantially harder to establish the corresponding upper bounds, though. In this chapter we derive such upper bounds for each of the three arrival processes. The proofs rely on the techniques developed in Section 2.2: for the model with M/G/∞ driven arrivals we use a proof that is based on a change-of-measure, whereas for the shot-noise and Hawkes driven arrivals we rely on a large-deviations based argumentation.

8.2 M/G/∞ driven arrivals

We here consider the case of an M/G/∞ driven net cumulative claim process. New customers arrive according to a Poisson process with rate $\lambda > 0$, and stay i.i.d. times in the system, with $\beta(\alpha)$ the LST of a generic sojourn time J. The number of customers simultaneously present thus has the dynamics of a so-called M/G/∞ system, whose stationary distribution is Poisson with parameter $\lambda E J$; see Theorem B.4. While in the system, each customer behaves as follows: she generates i.i.d. claims with LST $b(\alpha)$ according to a Poisson process with rate ν, and the claim processes of individual customers are assumed independent of each other and of everything else. The claim arrival rate is thus following a stochastic process $\Lambda(t)$ that is proportional to the number of customers in an M/G/∞ queue; see Figure 8.1.

In the analysis below we let premiums be generated at a constant rate r (by the full population, being of fluctuating size, that is); in Remark 8.2 below we consider the simpler case of each customer generating a premium rate r while being in the system.

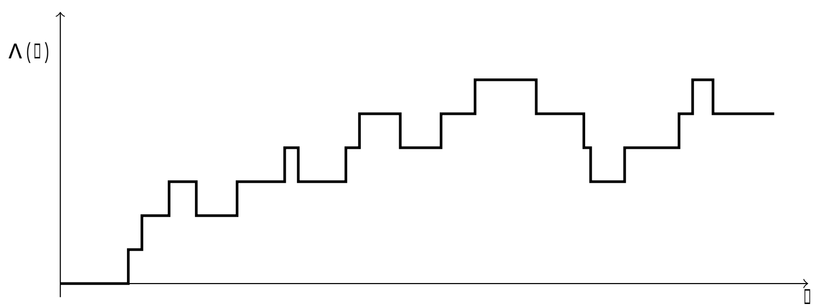

Fig. 8.1 Arrival rate process $\Lambda(t)$ in M/G/∞ case.

▷ *Limiting Laplace exponent.* In this case, the function $\phi(\alpha)$ can be found by an elementary calculation. As before, $X(t)$ is the net cumulative claim process. We impose the condition

$$\nu(\lambda E J)\, E B < r,$$

so that the process $X(t)$ eventually drifts to $-\infty$.

Proposition 8.1 *As $t \to \infty$,*

$$\frac{1}{t}\log E\, e^{-\alpha X(t)} \to \phi(\alpha) = r\alpha - \lambda + \lambda\beta(\nu(1 - b(\alpha))).$$

Proof. The number of customer arrivals in $[0,t)$ is Poisson with mean λt; it is a well-known property that, given the number of arrivals, each of them enters at a position that is uniformly distributed on $(0,t)$. We can therefore write

$$\frac{1}{t}\log \mathbb{E}\, e^{\vartheta Y(t)} = r\vartheta + \frac{1}{t}\log \sum_{k=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\,(\pi_t(\vartheta))^k = r\vartheta - \lambda + \lambda\,\pi_t(\vartheta),$$

where

$$\pi_t(\vartheta) := \frac{1}{t}\left(\int_0^t\int_0^u \mathbb{P}(J\in \mathrm{d}s)\sum_{j=0}^{\infty} e^{-\mu s}\frac{(\mu s)^j}{j!}\,(\beta(\vartheta))^j\,\mathrm{d}u + \int_0^t \mathbb{P}(J>u)\sum_{j=0}^{\infty} e^{-\mu u}\frac{(\mu u)^j}{j!}\,(\beta(\vartheta))^j\,\mathrm{d}u\right),$$

which simplifies to

$$\frac{1}{t}\left(\int_0^t\int_0^u \mathbb{P}(J\in \mathrm{d}s)\,e^{-\mu s(1-\beta(\vartheta))}\,\mathrm{d}u + \int_0^t \mathbb{P}(J>u)\,e^{-\mu u(1-\beta(\vartheta))}\,\mathrm{d}u\right);$$

here the first term corresponds to customers who have left by time t, whereas the second term corresponds to customers who are still present at time t. We are left with computing the limit of $\pi_t(\vartheta)$ as $t\to\infty$. Regarding the first term, it takes a minor computation to verify that, interchanging the integrals,

$$\frac{1}{t}\int_0^t\int_0^u \mathbb{P}(J\in \mathrm{d}s)\,e^{-\mu s(1-\beta(\vartheta))}\,\mathrm{d}u = \int_0^t \frac{t-s}{t}\,\mathbb{P}(J\in \mathrm{d}s)\,e^{-\mu s(1-\beta(\vartheta))}$$
$$\to \mathcal{J}(\mu(1-\beta(\vartheta))),$$

while the second term vanishes.

◦ *Change of measure.* Let $\vartheta^\star > 0$ solve $\kappa(\vartheta^\star) = 0$, which implicitly requires both the customers' sojourn times and claim sizes to have light-tailed distributions (see the argumentation in Section 2.2). We will prove the upper bound relying on a change-of-measure argument, similar to the one used in Section 2.2. Now the change of measure is such that $\kappa_{\mathbb{Q}}(\vartheta) = \kappa(\vartheta-\vartheta^\star)$. It is immediately checked that we can rewrite, with $\mu_{\mathbb{Q}} := \mu\beta(-\vartheta^\star)$ and $\mathcal{J}_{\mathbb{Q}} := \mathcal{J}(\mu-\mu_{\mathbb{Q}})$,

$$\begin{aligned}\kappa(\vartheta-\vartheta^\star) &= r(\vartheta-\vartheta^\star) - \lambda + \lambda\,\mathcal{J}(\mu(1-\beta(\vartheta-\vartheta^\star)))\\ &= r\vartheta - \lambda\mathcal{J}_{\mathbb{Q}} + \lambda\mathcal{J}_{\mathbb{Q}}\,\frac{\mathcal{J}\left(\mu_{\mathbb{Q}}\left(1-\dfrac{\beta(\vartheta-\vartheta^\star)}{\beta(-\vartheta^\star)}\right)+\mu-\mu_{\mathbb{Q}}\right)}{\mathcal{J}_{\mathbb{Q}}}.\end{aligned}$$

In other words, under the new measure $\mathbb{Q}$ the resulting exponentially twisted process is still an M/G/∞ driven net cumulative claim process, but now with customer arrival rate $\lambda_{\mathbb{Q}} := \lambda\,\mathcal{J}_{\mathbb{Q}}$, customer sojourn times with LST

$$\mathbb{E}_{\mathbb{Q}}\, e^{-\vartheta J} = \frac{\mathcal{J}(\vartheta+\mu-\mu_{\mathbb{Q}})}{\mathcal{J}(\mu-\mu_{\mathbb{Q}})}.$$

claim arrival rate _Q, and claim sizes with LST

$$\mathrm{E}_{\mathrm{Q}}4 \quad ^{U} = \frac{1^1U \quad \backslash ^{¢o}}{1^1 \quad \backslash ^{¢o}}•$$

Informally, the above means that the process . ¹C° reaching a high level D is the combined effect of: (i) a higher customer arrival rate, (ii) longer customer sojourn times, (iii) a higher claim arrival rate, and (iv) larger claims.

B *Uniform upper bound.* Our objective is to derive the upper bound ?¹D° 6 4 \¢D. We mimic the change-of-measure based approach of Section 2.2. At the moment g¹D° that »D- 1° has been reached, we have sampled the customer interarrival times ***F*** ¹ 1-•••- # ° and their sojourn times ***D*** ¹ 1-•••- # °. For each of the customers, we sample the number of claims during their sojourn time, i.e., ***M*** ¹ " 1-•••- " # °, where the corresponding arrival epochs are uniformly distributed over the sojourn time under consideration. The claim sizes are

B 11-•••- 1 " 1- 21-•••- 2 " 2-•••- # 1-•••-# " # •

The way we sample the path of the net cumulative claim process . ¹C° (until exceeding level D, that is) is the following.

1. We first generate the first customer arrival time, its sojourn time, the number of claims during its sojourn time, and the corresponding claim arrival times and claim sizes.
2. Then we sample whether the next customer arrival occurs before or after the first scheduled event (which is either a claim arrival or the departure of a customer). In the former case, we record the value of the net cumulative claim process at the first scheduled event. In the latter case, we sample this customer's arrival time (as well as its sojourn time, the number of claims during its sojourn time, and the claim arrival times and claim sizes).
3. We continue performing Step 2, until the current value of the net cumulative claim process exceeds D. Then we stop, which necessarily happens at a claim arrival.

Throughout this procedure, each time a random object is sampled, we update the likelihood ratio. We let g¹D°, as before, be the stopping time. We are to evaluate, adopting the approach that we have developed in Section 2.2, the expected (under Q, that is) likelihood ratio

$$\mathrm{E}_{\mathrm{Q}}\,!\,{}^1\boldsymbol{F}\text{-}\boldsymbol{D}\text{-}\boldsymbol{M}\text{-}\boldsymbol{B}^{\circ}\text{-}$$

which, bearing in mind that ${}^{0}_{\mathrm{Q}}{}^{1}0^{\circ}$ Ÿ 0, equals ?¹D°.

We let 5P¹ ° and 5Q¹ ° be the densities of under P and Q, respectively; likewise, 6P¹ ° and 6Q¹ ° are the densities of under P and Q, respectively. The likelihood ratio can be decomposed into four factors, as follows.

The first, which we will call ! L, corresponds to the customer arrivals. Suppose the first arrival is at time B, it leads to the evident contribution

$$\frac{\lambda}{\lambda_{\mathbb Q}}\frac{e^{-\lambda s}}{e^{-\lambda_{\mathbb Q} s}} = \frac{1}{\varrho_{\mathbb Q}}\frac{e^{-\lambda s}}{e^{-\lambda_{\mathbb Q} s}}.$$

For this customer we sample its specifics (sojourn time, claim arrival times, claim sizes). Regarding the next customer arrival, it is first checked whether this takes place before the next already scheduled event (which could be a claim arrival or a customer departure). More precisely, supposing that this next event is scheduled at (say) t time units from the current time, if this leads to a customer arrival at $s \in (0, t]$ time units from the current time, then we get a contribution to the likelihood ratio of

$$\frac{1-e^{-\lambda t}}{1-e^{-\lambda_{\mathbb Q} t}}\,\frac{\lambda e^{-\lambda s}/(1-e^{-\lambda t})}{\lambda_{\mathbb Q} e^{-\lambda_{\mathbb Q} s}/(1-e^{-\lambda_{\mathbb Q} t})} = \frac{\lambda}{\lambda_{\mathbb Q}}\frac{e^{-\lambda s}}{e^{-\lambda_{\mathbb Q} s}} = \frac{1}{\varrho_{\mathbb Q}}\frac{e^{-\lambda s}}{e^{-\lambda_{\mathbb Q} s}},$$

whereas if it does not lead to a customer arrival before the next scheduled event, then the contribution to the likelihood ratio is

$$\frac{e^{-\lambda t}}{e^{-\lambda_{\mathbb Q} t}}.$$

Combining the above, we find that, with N the number of customers that have arrived by time $\tau(u)$,

$$L_1 = e^{(\lambda_{\mathbb Q}-\lambda)\,\tau(u)}\left(\frac{\lambda}{\lambda_{\mathbb Q}}\right)^{N} = e^{(\lambda_{\mathbb Q}-\lambda)\,\tau(u)}\,\varrho_{\mathbb Q}^{-N}.$$

The second contribution, L_2, corresponds to the sojourn time durations. It is immediately verified that

$$L_2 = \prod_{i=1}^{N}\frac{g_{\mathbb P}(D_i)}{g_{\mathbb Q}(D_i)} = e^{(\nu-\nu_{\mathbb Q})\sum_{i=1}^{N} D_i}\,\varrho_{\mathbb Q}^{N}.$$

The third contribution concerns the claim arrival times. Observe that both under $\mathbb P$ and $\mathbb Q$, conditional on the number of arrivals, the arrival epochs are uniformly distributed over the interval, independently of each other (thus not contributing to the likelihood ratio). We therefore have the contribution, with $M_\bullet := \sum_{i=1}^{N} M_i$,

$$L_3 = \prod_{i=1}^{N}\frac{e^{-\nu D_i}(\nu D_i)^{M_i}/M_i!}{e^{-\nu_{\mathbb Q} D_i}(\nu_{\mathbb Q} D_i)^{M_i}/M_i!} = e^{(\nu_{\mathbb Q}-\nu)\sum_{i=1}^{N} D_i}\left(\frac{\nu}{\nu_{\mathbb Q}}\right)^{\sum_{i=1}^{N} M_i}$$
$$= e^{(\nu_{\mathbb Q}-\nu)\sum_{i=1}^{N} D_i}\,(1+\vartheta\,\beta'(\vartheta))^{-M_\bullet}.$$

The fourth and last contribution is due to the claim sizes. It is directly seen that this equals

$$L_4 = \prod_{i=1}^{N}\prod_{j=1}^{M_i}\frac{b_{\mathbb P}(B_{ij})}{b_{\mathbb Q}(B_{ij})} = e^{-\vartheta S_\bullet}\,(\beta(\vartheta))^{M_\bullet}, \quad\text{with}\quad S_\bullet := \sum_{i=1}^{N}\sum_{j=1}^{M_i} B_{ij}.$$

Since › is the sum of the claims generated by time g¹D°, we have

› Ag¹D° > . ¹g¹D°° ¡ D•

Using that inequality and recalling that A\¢ = a_Q a, we find an upper bound for ?¹D°:

?¹D° = E_Q ! ¹F- D- M- B° = E_Q ! L ! J ! S ! H
= 4^{¹a_Q a° g¹D°} 4 \¢ › 6 4 \¢D•

It can be seen as a Lundberg inequality for this M/G/1 driven counterpart of the classical Cramér-Lundberg model; observe that we have shown that the bound ?¹D° 6 4 \¢D applies uniformly for all D ¡ 0. In combination with the lower bound (see Remark 8.1), the following result is an immediate consequence.

Theorem 8.1 *In the model with* M/G/1 *driven arrivals,*

lim_{D ! 1} (1/D) log ?¹D° = \¢•

Remark 8.2 We here consider the variant in which each customer generates a premium at rate A while being in the system. This case turns out to be simpler than the one discussed above. An elementary calculation reveals that now

¹U° = a ¸ a 3 AU ¸ _¹1 1¹U°° = a ¸ a 3 i ¹U° •

In other words, equating this expression to 0 yields that \¢ should fulfil the familiar equation i ¹ \¢° = 0 pertaining to the system with a constant number of customers. This is not surprising: when = customers are present the (local) Laplace exponent is =AU =_¹1 1¹U°°, as both the claim arrival rate and the premium rate are proportional with =. Equating to 0 yields the \¢ that satisfies i ¹ \¢° = 0, irrespective of the value of =. In this model the process . ¹C° reaching a high level D is therefore essentially the combined effect of: (i) a higher claim arrival rate, and (ii) larger claims, i.e., the customer population dynamics do not play a role. ^

8.3 Shot-noise driven arrivals

In this section we consider the Cramér-Lundberg model in the situation that the arrival rate is a shot-noise process rather than a (constant-rate) Poisson process. Concretely, the (stochastic) arrival rate process ¹C° can be constructed as follows (cf. Figure 8.2). Let 8 be a sequence of i.i.d. non-negative random variables, distributed as the generic random variable with LST 3¹U° (where we remark that the random objects 8 and have a different meaning than in the previous section). Let " ¹C° be a Poisson process with intensity a ¡ 0, and let)8 be the 8-th arrival time that it generates. The parameter B ¡ 0 describes how fast the 'shots' decay in time, in that we can write

$$\lambda(t) = \sum_{i=1}^{N(t)} B_i\, e^{-r(t-\tau_i)}.$$

The main idea behind using a shot-noise arrival rate in the insurance context, is that the process is well suited to model the impact of (randomly arriving) catastrophic events. Indeed, events such as floods, windstorms, and earthquakes can be thought of as causing a 'pulse' in the claim arrival rate, which eventually fades away. Throughout we assume that

$$\frac{\mathbb{E}B}{r}\,\nu\,\mathbb{E}U < c,$$

so as to ensure that $Y(t)$ eventually drifts to $-\infty$.

◦ *Limiting Laplace exponent.* The number of claim arrivals in $[0,t]$ is mixed Poisson, i.e., Poisson with a random parameter $\Lambda(t)$, which is given by

$$\Lambda(t) := \int_0^t \lambda(s)\,\mathrm{d}s.$$

As before a crucial role is played by the solution $\alpha^\star$ of the equation $\varphi(\alpha) = 0$. So as to evaluate $\varphi(\alpha)$, observe that the above properties lead to

$$\begin{aligned}
\frac{1}{t}\log\mathbb{E}\,e^{\alpha Y(t)} &= -c\alpha + \frac{1}{t}\log\mathbb{E}\left(b(\alpha)\right)^{N(t)}\\
&= -c\alpha + \frac{1}{t}\log\mathbb{E}\left(\sum_{i=0}^{\infty} e^{-\Lambda(t)}\frac{(\Lambda(t))^i}{i!}\,(b(\alpha))^i\right)\\
&= -c\alpha + \frac{1}{t}\log\mathbb{E}\,e^{-\Lambda(t)(1-b(\alpha))}.
\end{aligned}$$

Hence to find an expression for $\varphi(\alpha)$, we are to compute the LST of $\Lambda(t)$. To this end, observe that we can write

$$\Lambda(t) = \sum_{i=1}^{N(t)} \int_0^{t-\tau_i} B_i\, e^{-rs}\,\mathrm{d}s = \sum_{i=1}^{N(t)} B_i\,\frac{1-e^{-r(t-\tau_i)}}{r}.$$

Recall that $N(t)$ has a Poisson distribution with parameter νt, and that, conditional on the number of shot arrivals, each of them arrives at a uniformly distributed epoch, independently of each other. Hence,

$$\begin{aligned}
\mathbb{E}\,e^{-\alpha\Lambda(t)} &= \sum_{k=0}^{\infty} e^{-\nu t}\frac{(\nu t)^k}{k!}\left(\int_0^t \frac{1}{t}\,\mathbb{E}\exp\left(-\alpha B_i\frac{1-e^{-rs}}{r}\right)\mathrm{d}s\right)^k\\
&= \exp\left(-\nu t + \nu\int_0^t \beta\left(\alpha\frac{1-e^{-rs}}{r}\right)\mathrm{d}s\right).
\end{aligned}$$

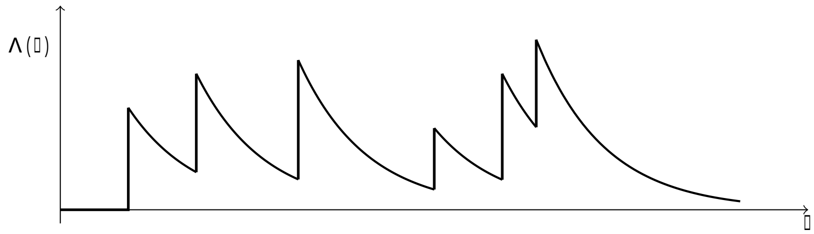

Fig. 8.2 Arrival rate process $\Lambda(t)$ in shot-noise case.

Upon combining the above, and sending t to ∞, we have proved the following result.

Proposition 8.2 *As $t \to \infty$,*

$$\frac{1}{t}\log \mathbb{E}\, e^{\vartheta \Lambda(t)} \to \kappa(\vartheta) := r\vartheta - \lambda\left(1 - \frac{1 - \beta(\vartheta)}{s}\right).$$

◦ *Upper bound.* To prove that the decay rate of $p(u)$ is upper bounded by $\vartheta^\star$, we use an adapted version of the large deviations based approach that was presented at the end of Section 2.2. The starting point is, for $u > r$, the upper bound

$$p(u) \leqslant \mathbb{P}(\exists n \in \mathbb{N} : Y(n) > u - r), \tag{8.2}$$

where it has been used that the net cumulative claim process decreases with at most r per unit of time. We have thus found an upper bound on $p(u)$ that corresponds to a *countable* number of events.

Recall the definition of $T^\star$ from Section 8.1, and the interpretation of $T^\star u$ being the typical time it takes the net cumulative claim process to exceed u. The intuition behind our proof is that we work with an upper bound on $p(u)$ that corresponds to a term that contains the contribution of epochs n in the order of $T^\star u$ (and that is therefore 'dominant'), and a term that contains the other contributions (and that is therefore 'negligible'). Indeed, applying (8.2) in combination with the union bound,

$$\begin{aligned} p(u) &\leqslant \sum_{n=1}^{T^\star(1+\varepsilon)u} \mathbb{P}(Y(n) > u - r) + \sum_{n=T^\star(1+\varepsilon)u+1}^{\infty} \mathbb{P}(Y(n) > u - r) \\ &\leqslant \sum_{n=1}^{T^\star(1+\varepsilon)u} \mathbb{P}(Y(n) > u - r) + \sum_{n=T^\star(1+\varepsilon)u+1}^{\infty} \mathbb{P}(Y(n) > 0), \end{aligned}$$

where $\varepsilon > 0$ will be picked below. We treat the two sums separately, starting with the second one.

As mentioned, the second sum does not contain contributions due to epochs n in the order of $T^\star u$, and should therefore be negligible relative to the first sum. Due to the Chernoff bound, for any \ ¡ 0 we have

$$\sum_{==)^{¢}¹1¸Yº D¸1} P¹.¹=º > 0º \; 6 \sum_{==)^{¢}¹1¸Yº D¸1} E4^{\backslash . ¹=º}•$$

Let \ ¡ 0 be such that ⁰¹ \ º = 0. Due to the fact that we are in the regime that there is a \¢ such that ¹ \¢º = 0, this \ exists (and is smaller than \¢); from ⁰¹0º ¡ 0 and ¹Uº being convex, we conclude that ¹ \ º Ÿ 0. It is readily seen that ¹ \ º = ¹0º ¡ 0; see Exercise 8.1. Let = be sufficiently large to ensure that

$$\frac{1}{=} \log E4^{\backslash . ¹=º} \; 6 \; ¹ \backslash º ¸ X = ¹0º ¸ X-$$

for some X 2 ¹0- ¹0ºº; this is possible as a consequence of Proposition 8.2. Recognizing a geometric sum, we thus end up with the upper bound, with I := exp¹ ¹0º ¸ Xº Ÿ 1,

$$\sum_{==)^{¢}¹1¸Yº D¸1} P¹.¹=º > 0º \; 6 \; \frac{I^{)^{¢}¹1¸Yº D¸1}}{1 \quad I}•$$

Hence the decay rate of the first sum is at most ¹ ¹0º Xº)¢¹1 ¸ Yº.

We now consider the first sum. This sum contains the most significant contributions, and will therefore turn out to be dominant. Again appealing to the Chernoff bound, and bounding a sum of non-negative terms by the product of the number of terms and the maximum summand,

$$\sum_{==1}^{)^{¢}¹Yº D} P¹.¹=º > D \quad Aº \; 6 \sum_{==1}^{)^{¢}¹Yº D} 4^{\backslash ¢¹D \; Aº} E4^{\backslash ¢ . ¹=º}$$

$$6 \;)^{¢}¹1 ¸ YºD \; 4^{\backslash ¢¹D \; Aº} \max_{==1-•••-)^{¢}¹1¸YºD} E4^{\backslash ¢ . ¹=º}•$$

Then observe that, using that the LST 3¹Uº is decreasing and 1 1¹ \¢º Ÿ 0, for any C > 0,

$$\log E4^{\backslash ¢ . ¹Cº} = A\backslash ¢C \quad aC ¸ a \int_0^C 3 \; ¹1 \quad 1¹ \backslash ¢ºº \frac{1 \quad 4^{BD}}{B} \, dD$$

$$6 \quad A\backslash ¢ \quad a ¸ a3 \; \frac{1 \quad 1¹ \backslash ¢º}{B} \quad C = ¹ \backslash ¢º C = 0• \qquad (8.3)$$

Upon combining the above, and using that D ¹ log D ! 0 as D ! 1, the decay rate of the first sum is at most \¢:

$$\lim_{D ! 1} \frac{1}{D} \log \quad)^{¢}¹1 ¸ YºD \; 4^{\backslash ¢¹D \; Aº} \max_{==1-•••-)^{¢}¹1¸YºD} E4^{\backslash ¢ . ¹=º} \; 6 \quad \backslash ¢•$$

We have thus obtained the upper bound, with the constant Y still to be chosen,

$$\lim_{D\,!\,1} \frac{1}{D} \log ?^1D^o \; 6 \quad \min \;\; \backslash^{¢}\text{-}{}^1\; {}^10^o \quad X^o)\; ^{¢\,1}1 \;\,¸\; Y^o \;\; •$$

We pick

$$Y \;\, ¡ \;\; \frac{\backslash^{¢}}{)^{¢}} \frac{1}{{}^10^o \quad X} \quad 1 = \frac{{}^11•)^{\;¢o}}{{}^10^o \quad X} \quad 1\text{-}$$

where the equality follows from $\backslash^{¢} =)^{¢}\;\; {}^11•)^{\;¢o}$; note that the number on the right-hand side is positive because ${}^10^o$ is increasing for $0 \;¡\; 0$. Under this choice $\backslash^{¢}\; Ÿ\; {}^1\; {}^10^o \quad X^o)\;^{¢\,1}1 \;¸\; Y^o$, so that the second sum asymptotically vanishes relative to the first sum. Combining this upper bound with the familiar lower bound, we have proven the following result.

Theorem 8.2 *In the model with shot-noise driven arrivals,*

$$\lim_{D\,!\,1} \frac{1}{D} \log ?^1D^o = \quad \backslash^{¢}•$$

Remark 8.3 At the expense of settling a few technical details, various extensions can be dealt with. In the first place, using the principles detailed in Section 2.4, the decay rate of $?^1D\text{-}DC^o$ (for given C and $D \;\;!\;\; 1$, that is) can be identified. In the second place, following a change-of-measure type of reasoning similar to the one used in Section 8.2, a uniform upper bound on $?^1D^o$ can be derived. Thirdly, where we now assumed the shots to decay exponentially, we can work with general decay functions: if the stochastic arrival rate has the form

$$^1C^o = \sum_{8=1}^{"^1C^o} \;\; {}_8 \;\; {}^1C \quad)_8{}^o\text{-}$$

for some non-negative function ${}^1D^o$, then the logarithmic asymptotics carry over, but now with the asymptotic Laplace exponent being

$$^1U^o = AU \quad a \;\; 1 \quad 3\;^{11}1 \quad 1^1U^{oo} \quad ^o \;\text{-} \qquad := \int_0^1 \;\; {}^1D^o \, \mathrm{d}D\text{-}$$

under the condition $\;\;E \quad a\,E \quad Ÿ\,A.$ ^

8.4 Hawkes driven arrivals

In this section the counting process $"\;^1C^o$, corresponding to the epochs $)_1\text{-}\,)_2\text{-}•••$ (meaning that the process $"\;^1C^o$ increases by 1 at $)_1\text{-}\,)_2\text{-}•••$), is defined as follows. Denote by ${}_8$ a sequence of i.i.d. non-negative random variables, distributed as the generic random variable with LST 3^1U^o (where we remark that the random objects ${}_8$ and have a different meaning than in the previous two sections). Let, as $C \;\#\; 0$,

$$\mathrm{P} \quad "\;^1C \;¸\quad C^o \quad "\;^1C^o = 1 \;j\quad {}^1B^o\text{-}\; B \;2\; »0\text{-}C¼ \;\; = \;\; {}^1C^o \quad C \;¸\; >^1 \;\; C^o\text{-}$$

and

$$\mathbb{P}\big(N(t+\Delta t) - N(t) = 0 \,|\, \Lambda(s), s\in[0,t]\big) = 1 - \Lambda(t)\,\Delta t + o(\Delta t),$$

where, for a given parameter $\lambda > 0$,

$$\Lambda(t) = \lambda + \sum_{i=1}^{N(t)} h(t - t_i) = \lambda + \sum_{i:\,t_i\leq t} h(t-t_i), \tag{8.4}$$

with $h(\cdot)$ a non-negative function. We call the process $\Lambda(t)$ a *Hawkes process.* See Figure 8.3 for a sample path of $\Lambda(t)$. The process, in which the current arrival rate depends on the observed sequence of past arrival times, is often referred to as being 'self-exciting'. In the insurance context a Hawkes arrival rate is used in case one wishes to model the effect of claims triggering additional claims.

In this section we consider the Cramér-Lundberg model with the time-varying, random arrival rate $\Lambda(t)$. We impose the condition $\|h\| < 1$, with

$$\|h\| := \int_0^\infty h(u)\,\mathrm{d}u,$$

under which the arrival rate process $\Lambda(t)$ does not explode as $t\to\infty$. In addition, we require that

$$\frac{1}{1-\|h\|}\,\lambda\,\mathbb{E}B < r,$$

under which $Y(t)$ eventually drifts to $-\infty$.

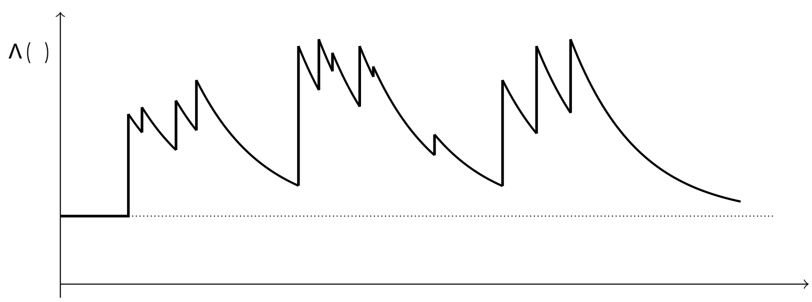

Fig. 8.3 Arrival rate process $\Lambda(t)$ in Hawkes case.

▷ *Limiting Laplace exponent, branching representation.* Under the above conditions, the next result provides us with an implicit characterization of the limiting Laplace exponent.

Proposition 8.3 *As* $t\to\infty$,

$$\frac{1}{t}\log \mathbb{E}\, e^{\alpha\, Y(t)} \to \varphi(\alpha) = r\alpha - \lambda\,(1-\beta(b(\alpha))),$$

where $\beta(z)$ is the unique root in $[0,1)$ *of*

$$\beta(z) = z\, d((1-\beta(z))). \tag{8.5}$$

The derivation of this result relies heavily on the representation of a Hawkes process as a *branching process*. To this end, observe that Equation (8.4) reveals that the Hawkes arrival process can be split into (i) a Poisson process with constant rate λ, in the sequel referred to as *immigrants*, and (ii) arrivals that are induced by the immigrants. Concretely, each of the immigrants increases the future arrival rate. The arrivals that occur due to this increase, are called the *children* of the immigrant. In turn, those children are potentially the parents of a next generation, and so forth. It is this recursive structure that we will exploit.

Proof (of Proposition 8.3). Recall that $N(t)$ denotes the number of claim arrivals in $[0,t)$. Our first objective is to analyze the transform of $N(t)$. Let $S(u)$ represent the number of children of an immigrant, u time units after its own birth, including the immigrant itself. Define the probability generating function $\beta(u,z) := \mathbb{E}\, z^{S(u)}$, for $z \in [0,1]$. Then

$$\mathbb{E}\, z^{N(t)} = \sum_{k=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\left(\frac{1}{t}\int_0^t \beta(u,z)\,\mathrm{d}u\right)^k = \exp\left(-\lambda t + \lambda\int_0^t \beta(u,z)\,\mathrm{d}u\right).$$

We are thus left with the identification of $\beta(u,z)$, which we do by studying each cluster separately. The key element of our argumentation is the distributional equality, for a fixed $t > 0$ and $u \in [0,t]$,

$$S(u) \stackrel{\rm d}{=} 1 + \sum_{i:T_i \leqslant u} S_i(u - T_i) = 1 + \sum_{i=1}^{\Pi(u)} S_i(u - T_i), \tag{8.6}$$

where the $S_i(u)$ are i.i.d. copies of $S(u)$; here $T_1, T_2, \dots$ are the birth times of the corresponding children, and $\Pi(u)$ is an inhomogeneous Poisson counting process with rate $\mu(u)$ (conditional on the sampled value of B that corresponds to the immigrant under consideration, that is). Here $S_i(u)$ can be interpreted as the number of children of child i (including the child itself).

Now denote by $P_t(s)$ the probability that, conditional on the fact that a child was born before time t, it was actually already born before time s, for $s \leqslant t$. Then it holds that

$$P_t(s) = \frac{\mathbb{P}(\Pi(s) = \Pi(t) = 1)}{\mathbb{P}(\Pi(t) = 1)} = \frac{\mathbb{P}(\Pi(s) = 1, \Pi(t) - \Pi(s) = 0)}{\mathbb{P}(\Pi(t) = 1)}.$$

Conditional on B, we thus find, with

$$A(s,t) := \int_s^t \mu(u)\,\mathrm{d}u,\quad M(t) := \int_0^t \mu(u)\,\mathrm{d}u,$$

that

$$\%_C{}^1B^o = \frac{A^10\text{-}B^o4\ ^{A^10\text{-}B^o}\ \ 4\ ^{A^1B\text{-}C^o}}{A^10\text{-}C^o4\ ^{A^10\text{-}C^o}} = \frac{{}^1B^o}{{}^1C^o};$$

note that cancels. Now define $?_C{}^1B^o := \%^0_C{}^1B^o = {}^1B^o\bullet\ {}^1C^o$. Appealing to the distributional equality (8.6) and conditioning on ,

$$
\begin{aligned}
[{}^1D\text{-}I^o &= \int_0^1 \sum_{:=0}^{1} \tilde{\text{Ö}}\ \mathrm{E}\ I^{({}^1D^o}\ \mathrm{j}\ \ {}^1D^o = :\text{-}\ \ = G\ \ \mathrm{P}^1\ \ {}^1D^o = :\ \mathrm{j}\ \ = G^o\,\mathrm{P}^1\ \ 2\,\mathrm{d}G^o\\
&= I\int_0^1 \sum_{:=0}^{1} \tilde{\text{Ö}}\ \mathrm{E}\ \text{"}\ \prod_{8=1}^{\ddot{\text{O}}} I^{({}^1D\ )_8^o}\ \#\ 4\ ^{G\ \ {}^1D^o}\frac{{}^1G\ \ {}^1D^{oo}:}{:!}\ \mathrm{P}^1\ \ 2\,\mathrm{d}G^o\\
&= I\int_0^1 \sum_{:=0}^{1} \tilde{\text{Ö}}\ \int_0^{D} [{}^1D\ \ B\text{-}I^o\ ?_D{}^1B^o\,\mathrm{d}B\ ^{:}\ 4\ ^{G\ \ {}^1D^o}\frac{{}^1G\ \ {}^1D^{oo}:}{:!}\ \mathrm{P}^1\ \ 2\,\mathrm{d}G^o\\
&= I\int_0^1 \exp\ \ G\int_0^{D} {}^11\ \ [{}^1D\ \ B\text{-}I^{oo}\ \ {}^1B^o\,\mathrm{d}B\ \ \mathrm{P}^1\ \ 2\,\mathrm{d}G^o\text{-}
\end{aligned}
$$

which leads to the fixed-point equation

$$[{}^1D\text{-}I^o = I\,3\int_0^{D} {}^11\ \ [{}^1D\ \ B\text{-}I^{oo}\ \ {}^1B^o\,\mathrm{d}B\ \bullet \tag{8.7}$$

Now we focus on identifying ${}^1U^o$. To this end, we first consider $\mathrm{E}\,4\ ^{U\cdot\ {}^1C^o}$, which we express in terms of $[{}^1D\text{-}1^1U^{oo}$. Observe that

$$\frac{1}{C}\log \mathrm{E}\,4\ ^{U\cdot\ {}^1C^o} = AU\ \ \frac{1}{C}\log \mathrm{E}\ \ 1^1U^{o\,\#\,{}^1C^o}\ \text{-}$$

so that

$$
\begin{aligned}
{}^1U^o &= AU\ \ a\ \ 1\ \ \lim_{C\,!\,1}\frac{1}{C}\int_0^{C}[{}^1D\text{-}1^1U^{oo}\,\mathrm{d}D\\
&= AU\ \ a^11\ \ [{}^11\text{-}1^1U^{ooo}\text{-}
\end{aligned}
$$

where, by virtue of (8.7), $[{}^11\text{-}I^o = [{}^1I^o$ solves (8.5).

B *Upper bound.* Using Proposition 8•3, we can now derive the upper bound on the decay rate of $?^1D^o$, leading to the following theorem.

Theorem 8.3 *In the model with Hawkes driven arrivals,*

$$\lim_{D\,!\,1}\frac{1}{D}\log ?^1D^o = \ \ \backslash^{\text{¢}}\bullet$$

Proof. The proof is completely analogous to that of Theorem 8.2, with the only issue being that we have to find the counterpart of Inequality (8.3). Recall that, from the very definition of $({}^1D^o$, it follows that $({}^11^o > ({}^1D^o$ for all $D > 0$, so that for all $I\ \ ¡\ \ 1$

we have that $[^1D\text{-}I^{\circ}\ 6\ [^1 1\text{-}I^{\circ}$. Using that $1^1\ \ \backslash^{\text{¢}\circ}\ ¡\ 1$, we consequently find that

$$\int_0^1 [^1D\text{-}1^1\ \ \backslash^{\text{¢}\circ\circ}\,\mathrm{d}D\ 6\ \int_0^1 [^1 1\text{-}1^1\ \ \backslash^{\text{¢}\circ\circ}\,\mathrm{d}D = C\,[^1 1^1\ \ \backslash^{\text{¢}\circ\circ}.$$

Hence,

$$\begin{aligned}\log \mathrm{E}\,4^{\backslash^{\text{¢}}\cdot\ ^1C^{\circ}} &= \ A\backslash^{\text{¢}}C\ \ \ aC\ _{\text{¸}}\ a\int_0^1 [^1D\text{-}1^1\ \ \backslash^{\text{¢}\circ\circ}\,\mathrm{d}D\\ &6\ \ \ A\backslash^{\text{¢}}\ \ \ a\ 1\ \ \ [^1 1^1\ \ \backslash^{\text{¢}\circ\circ}\ \ \ C =\ \ \ ^1\ \ \backslash^{\text{¢}\circ}C = 0\text{-}\end{aligned}$$

which completes the proof.

8.5 Discussion and bibliographical notes

This chapter has concentrated on variants of the Cramér-Lundberg model in which arrivals tend to cluster, unlike in the conventional Poisson-driven Cramér-Lundberg model. Three natural models have been discussed, in each of which the arrival rate is described by a stochastic process: an M/G/1 queue, a shot-noise process, and a Hawkes process. Needless to say that various alternative arrival rate processes could be thought of.

It can be seen that neglecting the variability of the arrival rate leads to an underestimation of the ruin probability. Consider for instance the model in which the claim arrival rate is described by an M/G/1 queue; similar considerations apply to the models with shot-noise and Hawkes claim arrival rates. Using that the convexity of LST s entails that $3^1U^{\circ} > 1\ \ \ U\,\mathrm{E}$, we find that, for any U,

$$\begin{aligned}{}^1U^{\circ} &= AU\ \ \ a\ _{\text{¸}}\ a\,3\ \_^1 1\ \ \ 1^1U^{\circ\circ}\ \ > AU\ \ \ a\ _{\text{¸}}\ a\ 1\ \ \ \_^1 1\ \ \ 1^1U^{\circ\circ}\,\mathrm{E}\\ &= AU\ \ \ a\,\mathrm{E}\ \ \ \_^1 1\ \ \ 1^1U^{\circ\circ}\text{-}\end{aligned}$$

where the last expression can be interpreted as the Laplace exponent $_{\text{Pois}}{}^1U^{\circ}$ of the conventional Cramér-Lundberg model with claim arrival rate $a\,\mathrm{E}\ \ \_$ (where the premium rate is A, and claims have LST 1^1U°). One indeed obtains that the decay rate $\backslash^{\text{¢}}$ is smaller than its counterpart, say $\backslash^{\text{¢}}_{\text{Pois}}$, in the corresponding Cramér-Lundberg model, as illustrated by Figure 8.4.

The model considered in Section 8.2, with a fluctuating number of customers, resembles multiple timescale models analyzed in the queueing context. We refer to e.g. the large deviations analysis presented in [16] (albeit focusing on an other scaling regime than the one we considered in Section 8.2). The Lundberg-type upper bound, based on a change-of-measure, is novel (but strongly mimics the line of reasoning used in Section 2.2).

Shot-noise [3] as such has been covered in Chapter 6; shot-noise as the mechanism underlying the arrival rate in a ruin model, being the topic of Section 8.3, has been explored substantially less, however. An account of shot-noise driven ruin models,

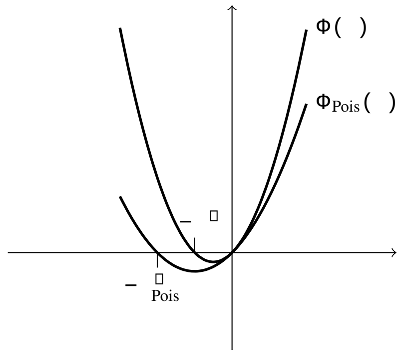

Fig. 8.4 The functions (U) (top curve) and ${}_{\rm Pois}(U)$ (bottom curve). The ordering of the decay rates is indicative of the fact that ruin is more likely in the model with a variable arrival rate.

with early references in this area being [13, 4], is presented in [2, Section XIII.4], which is partly based on [1]. Our proof of the decay rate of the ruin probability was inspired by the approach developed in [7]. Where we focused on the decay rate, uniform upper bounds can be derived as well; see e.g. [17]. The analysis of Section 8.3 covers the asymptotics in the light-tailed regime; results for the heavy-tailed case can be found in e.g. [1, 19].

The Hawkes process, which in Section 8.4 serves as the claim arrival process, has been developed in the early 1970s [8, 9, 10]; for more background on Hawkes processes at large we refer to [15]. The proof of Proposition 8•3 is based on the argumentation used in [14, Theorem 4.1]; the proof of [14, Theorem 4.1] also covers the uniqueness of a solution $[(I)\ 2\ [0\text{-}\ 1)$ in (8.5). We refer to e.g. [11] for a detailed account of the cluster representation and connections with branching processes, underlying the proof of Proposition 8•3. Early studies on (infinite-server) queues fed by a Hawkes driven arrival process are [5, 14], where their multivariate counterpart is covered by [12]. A proof of Theorem 8.3, for the special case that the shots are deterministic, has been provided in [18]; results for the heavy-tailed case are given in [19].

While this chapter focuses on the setting of the claim size distribution being light-tailed, one could consider the subexponential counterpart as well. In this context it is anticipated, in accordance with the 'principle of a single big claim' discussed in Section 2.3, that, for large D, ruin is with overwhelming probability due to one exceptionally large claim.

Exercises

8.1 Let the Legendre transform (0) be as defined in (8.1).
(i) Prove that (0) is convex.

(ii) Prove that ${}^{1}0^{\text{o}} > 0$ for all 0, and that ^{1} ${}^{0}{}^{1}0^{\text{oo}} = 0$. Prove that for $0 >$ ${}^{0}{}^{1}0^{\text{o}}$, we can restrict ourselves to optimizing over $\backslash > 0$. Use this to show that if ${}^{0}{}^{1}0^{\text{o}}$ Ÿ 0, then ${}^{1}0^{\text{o}}$ is increasing on »0- 1º.
(iii) Let $\backslash$ be such that ${}^{0}{}^{1}$ $\backslash$ º = 0. Prove that ${}^{1}0^{\text{o}} =$ ^{1} $\backslash$ º.

8.2 Consider the model with M/G/1 driven arrivals. Let the sojourn time be exponentially distributed with parameter b ¡ 0 and let the claim size be exponentially distributed with parameter `.
(i) Prove that the net-profit condition, which is

$$ {}^{0}{}^{1}0^{\text{o}} = \lim_{C\,!\,1} \frac{\mathrm{E}\,.\,{}^{1}C^{\text{o}}}{C} \;\text{Ÿ}\; 0 $$

in this context, is equivalent with _a Ÿ Ab `.
(ii) Determine ${}^{1}U^{\text{o}}$. Prove that the net cumulative claim process is of Cramér-Lundberg type, with premium rate A, claim arrival rate a, and claim sizes with LST

$$ 1\ {}^{1}U^{\text{o}} := \frac{b}{b\ \text{¸}\ \_}\ \text{¸}\ \frac{\_}{b\ \text{¸}\ \_}\,\frac{Z}{Z\ \text{¸}\ U}- $$

with Z := b `•${}^{1}b$ ¸ _º. Interpret the distribution corresponding to 1 ${}^{1}U^{\text{o}}$.
(iii) Show that

$$ \backslash^{\text{¢}} = \frac{b}{b\ \text{¸}\ \_}\,\text{`}\quad \frac{\_}{b\ \text{¸}\ \_}\,\frac{a}{A}\ \text{¡}\ 0\text{•} $$

(iv) We now consider the model under Q• Show that

$$ \_{}_{\mathrm{Q}} = {}^{1}b\ \text{¸}\ \_{}^{\text{o}}\frac{\text{`}A}{a\ \text{¸}\ \text{`}A}- \qquad a_{\mathrm{Q}} = \frac{b}{b\ \text{¸}\ \_}\,{}^{1}a\ \text{¸}\ \text{`}A^{\text{o}}\text{•} $$

Show that under Q both and are exponentially distributed, with parameters

$$ \text{`}_{\mathrm{Q}} = \frac{\_}{b\ \text{¸}\ \_}\,\frac{a\ \text{¸}\ \text{`}A}{A}- \qquad b_{\mathrm{Q}} = {}^{1}b\ \text{¸}\ \_{}^{\text{o}}\frac{a}{a\ \text{¸}\ \text{`}A}\text{•} $$

(v) Show that under Q the process has a positive drift:

$$ \lim_{C\,!\,1} \frac{\mathrm{E}_{\mathrm{Q}}\,.\,{}^{1}C^{\text{o}}}{C} = A\quad \frac{Ab\text{`}}{\_a}\quad 1\quad \text{¡}\ 0\text{•} $$

8.3 Consider the model with shot-noise driven arrivals. Prove that ${}^{1}U^{\text{o}} >$ ${}_{\text{Pois}}{}^{1}U^{\text{o}}$, where ${}_{\text{Pois}}{}^{1}U^{\text{o}}$ denotes the Laplace exponent of the conventional Cramér-Lundberg model with claim arrival rate a E •B (where the premium rate is A, and claims have LST $1{}^{1}U^{\text{o}}$).

8.4 Consider the model with Hawkes driven arrivals.
(i) Use (8.5) to prove that

$$ \mathrm{E}\,(\,{}^{1}1^{\text{o}} = \frac{1}{1\quad \mathrm{E}}\text{•} $$

Also determine Var $($ ${}^{1}1^{\text{o}}$•

(ii) Argue that, for $I > 0$, and a random variable $- 2$ N,

$$\mathrm{E}\, I^{-} > 1 \,\text{¸}\, ^{1}I \quad 1^{\circ}\, \mathrm{E}\, - \bullet$$

(iii) Use (i) and (ii) to prove that $^{1}U^{\circ} > \ _{\text{Pois}}{}^{1}U^{\circ}$, where $_{\text{Pois}}{}^{1}U^{\circ}$ denotes the Laplace exponent of the conventional Cramér-Lundberg model with claim arrival rate

$$\frac{a}{1 \quad \mathrm{E}}$$

(where the premium rate is A, and claims have LST $1^{1}U^{\circ}$).

8.5 In this exercise we formally verify the *steepness* condition, needed to legitimately apply the Gärtner-Ellis theorem, for a special instance of the model with Hawkes driven arrivals. Define by the set of $\backslash$ for which $^{1} \ \backslash^{\circ}\ \ddot{Y}\ 1$, and m its boundary. Then steepness means that

$$\lim_{\backslash \, " m} \ {}^{01} \ \backslash^{\circ} = 1\bullet$$

We consider the Hawkes driven arrival model with 1. Assume $\ddot{Y}$ 1 and $a\mathrm{E} \ \ddot{Y} A^{1}1 \quad {}^{\circ}$ to make sure that the Hawkes process does not explode and that $.\, ^{1}C^{\circ}$ eventually drifts to $1\bullet$

(i) Use (8.5) to prove that

$$[^{01}I^{\circ} = \frac{[^{1}I^{\circ}}{I} \,\text{¸}\, [^{1}I^{\circ}\, [^{01}I^{\circ} \quad \bullet$$

(ii) Prove that there is a $I_0 \ ¡ \ 1$ for which $[^{1}I_0{}^{\circ} = 1\bullet$, that $[^{01}I_0{}^{\circ} = 1$ (the derivative to be understood as the left derivative), and $[^{1}I^{\circ} = 1$ for $I \ ¡ \ I_0\bullet$ (*Hint*: use that $[^{1}I^{\circ}$ is increasing and convex for $I > 1$, with $[^{1}1^{\circ} = 1$ and $[^{01}1^{\circ} \ ¡ \ 1$.)

(iii) Conclude that $^{1} \ \backslash^{\circ}$ is steep in the $\backslash$ for which $1^{1} \ \backslash^{\circ} = I_0$ (assumed to exist), in that

$$\lim_{\backslash \, " \, 1 \ ^{11}I_0{}^{\circ}} \ {}^{01} \ \backslash^{\circ} = 1\bullet$$

8.6 $^{1}¢^{\circ}$ Consider the model with Hawkes arrivals. Let, as before, Q be such that the limiting Laplace exponent equals $_{\mathrm{Q}}{}^{1}U^{\circ} = \ ^{1}U \quad \backslash^{¢\circ}$, with $\backslash^{¢} \ ¡ \ 0$ such that $^{1} \ \backslash^{¢\circ} = 0$. In this exercise we will show that $_{\mathrm{Q}}{}^{1}U^{\circ}$ corresponds to a model with Hawkes arrivals, too, but with different parameters and distributions.

(i) Recall that $^{1}U^{\circ} = AU \quad a^{1}\,1 \quad [^{1}1^{1}U^{\circ\circ\circ}$. In this part of the exercise we focus on determining a, $1^{1}U^{\circ}$ and $[^{1}I^{\circ} = \mathrm{E}\, I^{(}$ under Q, which we denote by a_{Q}, $1_{\mathrm{Q}}{}^{1}U^{\circ}$, and $[_{\mathrm{Q}}{}^{1}I^{\circ}$. Show that

$$a_{\mathrm{Q}} = a\, [^{1}1^{1} \ \backslash^{¢\circ\circ}\text{-} \quad 1_{\mathrm{Q}}{}^{1}U^{\circ} = \frac{1^{1}U \quad \backslash^{¢\circ}}{1^{1} \ \backslash^{¢\circ}}\text{-} \quad [_{\mathrm{Q}}{}^{1}I^{\circ} = \frac{[^{1}I\, 1^{1} \ \backslash^{¢\circ\circ}}{[^{1}1^{1} \ \backslash^{¢\circ\circ}}\bullet$$

Argue that this means that

$$\mathrm{Q}^{1} \quad 2\, \mathrm{d}G^{\circ} = \mathrm{P}^{1} \quad 2\, \mathrm{d}G^{\circ} \frac{4^{\backslash^{¢}G}}{1^{1} \ \backslash^{¢\circ}}\text{-} \quad \mathrm{Q}^{1}(\,=\, =^{\circ} = \mathrm{P}^{1}(\,=\, =^{\circ} \frac{^{1}1^{1} \ \backslash^{¢\circ\circ}=}{[^{1}1^{1} \ \backslash^{¢\circ\circ}}\bullet$$

(ii) Recall that $\lceil^1 l^{\circ}$ is the unique root in »0- 1° of the fixed point equation $\lceil^1 l^{\circ} = l\, 3^{11}1 \quad \lceil^1 l^{\circ\circ} \quad {}^{\circ}$• In this part we determine $3^1 U^{\circ}$ and ${}^1 \complement^{\circ}$ under Q (i.e., corresponding to $\lceil_{\mathrm{Q}}{}^1 l^{\circ}$, as determined in part (i)), which we denote by $3_{\mathrm{Q}}{}^1 U^{\circ}$ and ${}_{\mathrm{Q}}{}^1 \complement^{\circ}$. Show that

$$3_{\mathrm{Q}}{}^1 U^{\circ} = \frac{3^1 U \,\text{¸}\, {}^1 1 \quad \lceil^1 1^1 \quad \backslash^{\text{¢}\circ\circ\circ} \quad {}^{\circ}}{3^{11}1 \quad \lceil^1 1^1 \quad \backslash^{\text{¢}\circ\circ\circ} \quad {}^{\circ}} - \qquad {}_{\mathrm{Q}}{}^1 \complement^{\circ} = {}^1 \complement^{\circ} \lceil^1 1^1 \quad \backslash^{\text{¢}\circ\circ}•$$

Argue that this means that

$$\mathrm{Q}^1 \quad 2\, \mathrm{d}G^{\circ} = \mathrm{P}^1 \quad 2\, \mathrm{d}G^{\circ} \frac{4^1 \lceil^1 1^1 \quad \backslash^{\text{¢}\circ\circ} \quad 1^{\circ} \quad G}{3^{11}1 \quad \lceil^1 1^1 \quad \backslash^{\text{¢}\circ\circ\circ} \quad {}^{\circ}}•$$

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Chapter 9
Dependence between claim sizes and interarrival times

Abstract — Where in the conventional Cramér-Lundberg model claim sizes and claim interarrival times are independent, this chapter studies the case in which this independence assumption is lifted. We first consider two models in which the claim size is correlated with the previous claim interarrival time, and subsequently two models in which the claim interarrival time is correlated with the previous claim size. In all models the double transform pertaining to the time-dependent ruin probability is computed.

9.1 Introduction

In the classical Cramér-Lundberg model, claim sizes and claim interarrival times are assumed to be independent. In recent years there has been a growing interest in the, often realistic, case in which claim sizes and claim interarrival times exhibit some form of correlation. The present chapter discusses various model variants that incorporate such dependence.

In Section 9.2 we analyze two models in which a claim size, through a specific mechanism, is correlated with the previous interclaim time. In the first model, every claim size $_=$ provides the parameter of the exponential distribution that defines the preceding interarrival time. In the second model, every claim size $_=$, *together* with an independently sampled random variable $+_=$, determines that arrival rate.

In Section 9.3 we consider a model in which a claim interarrival time is correlated with the *previous* claim size. In this case, the arrival rate is allowed to attain only a finite set of possible values. In Section 9.4 we discuss a model in which again the interarrival times depend on previous claim sizes, but we allow additional flexibility compared to the model of Section 9.3. For each of these models, we compute the double transform of the time-dependent ruin probability. We use techniques resembling those used earlier in this monograph: we analyze the time-dependent ruin probability at an exponentially distributed time horizon $)_V$, and in addition transform with respect to the initial reserve level D. At a high level, the approach followed in Section 9.4 is the same as the one relied upon in Section 9.3, but a more delicate analysis is required (in particular when proving that a certain equation has precisely 3 zeroes, which is needed to determine the 3 unknowns).

9.2 Claim size being correlated with previous interarrival time

In this section we analyze two models with correlation between claim size and the previous interclaim time. Model 1 will be analyzed by applying Method 1 presented in Section 1.3, while for Model 2 we (essentially) rely on Method 4 of Section 1.6. Remarkably, while Model 2 is more general than Model 1, the analysis for Model 2 is more explicit. This illustrates that the methods introduced in Chapter 1 are not equivalent: for specific variants of the Cramér-Lundberg model one method can be better suited than another one.

▷ *Model 1: mechanism and dependence structure.* In this model, the claim size directly determines the parameter of the exponential distribution of the preceding interclaim time. Model 2 (that is subsequently treated) is slightly more involved, considering a less direct relation between claim size and rate of the preceding interclaim time; it can be seen as a counterpart of the model that will be discussed in Section 9.3, where we consider correlation between the interclaim time and the *previous* claim size.

We proceed by describing the underlying dependence mechanism of Model 1. After a claim arrival, the next interarrival time and corresponding claim size are chosen as follows: if the claim size has value $x > 0$, then the length of the interval between the previous claim's arrival time and this claim's arrival time has an exponential distribution with parameter $\lambda(x) > 0$. The time-dependent ruin probability $p(u,t)$ and the double transform $\pi(\alpha,\beta)$ are defined in the usual manner.

▷ *Model 1: transform of the ruin probability.* It is readily checked that the approach set up in Section 1.3 carries over. This means that $\pi(\alpha,\beta)$ can be written as the sum of $\pi_1(\alpha,\beta)$ and $\pi_2(\alpha,\beta)$, where the former reflects the contribution corresponding to ruin due to the first arriving claim, and the latter to ruin occurring later. Regarding the first contribution, we thus obtain

$$\pi_1(\alpha,\beta) = \int_0^\infty \frac{\lambda(x)}{\lambda(x)+\beta}\left(\frac{1-e^{-\alpha x}}{\alpha} - \frac{e^{-(\lambda(x)+\beta)x/c} - e^{-\alpha x}}{\alpha - (\lambda(x)+\beta)/c}\right)\mathbb{P}(B\in \mathrm{d}x),$$

as justified by Equation (1.3). With $\gamma(x,\beta)$ defined as $(\lambda(x)+\beta)/c$, this quantity can be interpreted as

$$\pi_1(\alpha,\beta) = \mathbb{E}\left(\frac{\lambda(B)}{\lambda(B)+\beta}\left(\frac{1-e^{-\alpha B}}{\alpha} - \frac{e^{-\gamma(B,\beta)B} - e^{-\alpha B}}{\alpha - \gamma(B,\beta)}\right)\right).$$

which is an object that we can in principle calculate (as we know the distribution of the generic claim size B). Regarding the second contribution, we obtain by following the line of reasoning of Section 1.3,

$$\phi_2(\alpha,\beta) = \int_0^\infty \frac{\lambda(x)}{r} \int_x^\infty \frac{e^{-s(x,\beta)y} - e^{-\alpha y}}{\alpha - s(x,\beta)}\, p(y,x,f_\beta)\,\mathrm{d}y\; \mathbb{P}(B \in \mathrm{d}x).$$

It is directly seen that this $\phi_2(\alpha,\beta)$ can be written as the difference of

$$\begin{aligned}
\phi_2^{+}(\alpha,\beta) &:= \int_0^\infty \frac{\lambda(x)}{r} \int_x^\infty \frac{e^{-s(x,\beta)y}}{\alpha - s(x,\beta)}\, p(y,x,f_\beta)\,\mathrm{d}y\; \mathbb{P}(B \in \mathrm{d}x)\\
&= \int_0^\infty \frac{\lambda(x)}{r(\alpha - s(x,\beta))} e^{-s(x,\beta)x} \int_0^\infty e^{-s(x,\beta)y}\, p(y,f_\beta)\,\mathrm{d}y\; \mathbb{P}(B \in \mathrm{d}x)\\
&= \mathbb{E}\left(\frac{\lambda(B)}{r(\alpha - s(B,\beta))} e^{-s(B,\beta)B}\, \phi(s(B,\beta),\beta)\right),
\end{aligned}$$

which is an expression that we cannot further evaluate (yet), and

$$\begin{aligned}
\phi_2^{-}(\alpha,\beta) &:= \int_0^\infty \frac{\lambda(x)}{r} \int_x^\infty \frac{e^{-\alpha y}}{\alpha - s(x,\beta)}\, p(y,x,f_\beta)\,\mathrm{d}y\; \mathbb{P}(B \in \mathrm{d}x)\\
&= \int_0^\infty \frac{\lambda(x)}{r(\alpha - s(x,\beta))} e^{-\alpha x}\, \mathbb{P}(B \in \mathrm{d}x) \int_0^\infty e^{-\alpha y}\, p(y,f_\beta)\,\mathrm{d}y\\
&= \mathbb{E}\left(\frac{\lambda(B)}{r(\alpha - s(B,\beta))} e^{-\alpha B}\right) \phi(\alpha,\beta).
\end{aligned}$$

Observe that

$$\bar\phi(\alpha,\beta) := \mathbb{E}\left(\frac{\lambda(B)}{r(\alpha - s(B,\beta))} e^{-\alpha B}\right) = \mathbb{E}\left(\frac{\lambda(B)}{\lambda(B) + \beta - r\alpha} e^{-\alpha B}\right),$$

which we can evaluate, as we know the distribution of B.

▷ *Model 1: solving the double transform.* Upon combining the above, we can isolate the quantity of our interest, i.e., $\phi(\alpha,\beta)$. Indeed, we obtain that

$$\phi(\alpha,\beta) = \frac{\phi_1(\alpha,\beta) + \phi_2^{+}(\alpha,\beta)}{1 + \bar\phi(\alpha,\beta)}. \tag{9.1}$$

Importantly, it should be realized that $\phi_2^{+}(\alpha,\beta)$ is not known yet. In the sequel we restrict ourselves to the case that the claim size distribution is given by

$$\mathbb{P}(B \leqslant x) = \sum_{i=1}^{d} p_i\, \mathbf{1}(x \geqslant t_i),$$

with $\mathbf{1}(\cdot)$ the unit step function and $p_1,\ldots,p_d \geqslant 0$, $\sum_{i=1}^{d} p_i = 1$. We then have d possible claim arrival rates $\lambda(t_1),\ldots,\lambda(t_d)$, where we assume (without loss of generality) that $\lambda(t_1) \leq \lambda(t_2) \leq \cdots \leq \lambda(t_n)$. Then

$$1 + \phi(s,\delta) = 1 - \sum_{i=1}^{n} p_i \frac{\lambda(t_i)}{\lambda(t_i) + \delta - cs} e^{-st_i} = \frac{f(s) - g(s)}{f(s)}, \tag{9.2}$$

where

$$f(s) := \prod_{i=1}^{n} (\lambda(t_i) + \delta - cs), \quad g(s) := \sum_{i=1}^{n} p_i \frac{\lambda(t_i)}{\lambda(t_i) + \delta - cs} e^{-st_i} f(s).$$

Application of Rouché's theorem (Theorem A.1) to the numerator of the last term in (9.2) reveals that it has exactly n zeroes in the right-half s-plane. Indeed, (i) $f(s)$ and $g(s)$ are analytic in that half-plane; (ii) $|f(s)| > |g(s)|$ on the imaginary axis and on the large semi-circle from $+iR$ to $-iR$ with center at the origin and radius R in that half-plane (and letting $R \to \infty$); and (iii) $f(s)$ has n zeroes in the right-half s-plane. Inspection of the behavior of $1 + \phi(s,\delta)$ at the asymptotes $s = \rho(t_i,\delta)$, $i = 1,\ldots,n$, reveals that these n zeroes of $1 + \phi(s,\delta)$, viz., $s_1^*(\delta),\ldots,s_n^*(\delta)$, are all real, exactly one being located in $(0,\rho(t_1,\delta))$, one in $(\rho(t_1,\delta),\rho(t_2,\delta))$, etc.

Since $\phi(s,\delta)$ is an analytic function of s in the right-half s-plane, the numerator in the right-hand side of (9.1) must be zero for $s = s_j^*(\delta)$, $j = 1,\ldots,n$. This results in n linear equations in the n remaining unknowns $\phi(\rho(t_j,\delta),\delta)$ featuring in $\phi_2^+(s,\delta)$. Thus, $\phi(s,\delta)$ is completely determined.

Remark 9.1 The above can be generalized to the following case: If $B = x \in [t_{j-1},t_j)$ then $\lambda(x) = \lambda(t_j)$, $j = 1,\ldots,n$ (with $t_0 = 0$ and $t_n = \infty$). In (9.2) one should now replace $p_i e^{-st_i}$ by

$$\int_{t_{i-1}}^{t_i} e^{-sx} \mathrm{P}(B \in \mathrm{d}x),$$

which does not complicate matters; and one still has n unknown functions $\phi(\rho(t_j,\delta),\delta)$ which follow via an application of Rouché's theorem (Theorem A.1). However, claim size distributions which give rise to a continuum of different $\lambda(\cdot)$ values seem to make the problem of determining the unknown function $\phi_2^+(s,\delta)$ much less tractable.

B *Model 2: mechanism and dependence structure.* We now introduce and analyze a very similar model in which the structure of the dependence between claim size and preceding interclaim time is slightly more involved and less direct. In this model, an exact analysis of the ruin probability is possible even for a continuum of possible interclaim arrival rates.

The sequence $B_1, B_2, \ldots$ represents the i.i.d. claim sizes. $\Theta_1, \Theta_2, \ldots$ is a second sequence of i.i.d. random variables, independent of the claim sizes. After the n-th claim arrival, a new claim interarrival time A_{n+1}, threshold value Θ_{n+1} and claim size B_{n+1} are drawn. If $B_{n+1} = x$ and $\Theta_{n+1} = x \cdot y$, then A_{n+1} is exponentially distributed with parameter $\lambda(y) > 0$.

▷ *Model 2: transform of the ruin probability.* Again our objective is to characterize the time-dependent ruin probability, with a time window that is exponentially distributed with parameter β. Denoting this probability by $p(u,T_\beta)$, we thus obtain, as $\Delta t\downarrow 0$,

$$\begin{aligned} p(u,T_\beta) &= \left(1-\int_0^\infty\int_0^\infty \lambda(s)\,\Delta t\,\mathbb{P}(B\in \mathrm{d}y)\,\mathbb{P}(y/X\in \mathrm{d}s)-\beta\,\Delta t\right)p(u+r\,\Delta t,T_\beta)\\ &+\int_0^\infty\int_u^\infty \lambda(s)\,\Delta t\,\mathbb{P}(B\in \mathrm{d}y)\,\mathbb{P}(y/X\in \mathrm{d}s)\\ &+\int_0^\infty\int_0^u \lambda(s)\,\Delta t\,\mathbb{P}(B\in \mathrm{d}y)\,\mathbb{P}(y/X\in \mathrm{d}s)\,p(u-y,T_\beta), \end{aligned} \tag{9.3}$$

up to $o(\Delta t)$ terms. Define

$$\varphi(\alpha):=\int_0^\infty e^{-\alpha y}\,\mathbb{P}(B\in \mathrm{d}y)\int_0^\infty \lambda(s)\,\mathbb{P}(y/X\in \mathrm{d}s)$$

(assuming that the claim arrival rate function $\lambda(\cdot)$ is such that $\varphi(\alpha)$ is well-defined). We proceed by following the standard procedure in Equation (9.3) (i.e., subtracting $p(u+r\,\Delta t,T_\beta)$ from both sides, dividing by Δt, and then taking the limit $\Delta t\downarrow 0$). We thus arrive at the differential equation

$$\begin{aligned} r\frac{\mathrm{d}}{\mathrm{d}u}p(u,T_\beta) = {}&(\varphi(0)+\beta)\,p(u,T_\beta)\\ &-\int_0^\infty\int_u^\infty \lambda(s)\,\mathbb{P}(B\in \mathrm{d}y)\,\mathbb{P}(y/X\in \mathrm{d}s)\\ &-\int_0^\infty\int_0^u \lambda(s)\,\mathbb{P}(B\in \mathrm{d}y)\,\mathbb{P}(y/X\in \mathrm{d}s)\,p(u-y,T_\beta). \end{aligned} \tag{9.4}$$

As before, the next step is to transform this identity with respect to u, i.e., we multiply both sides of Equation (9.4) by $e^{-\alpha u}$ and integrate over u. With $\pi(\alpha,\beta)$ defined in the usual manner, and with $\bar p(\beta):=p(0+,T_\beta)$, we end up with the following result.

Proposition 9.1 *For any $\alpha,\beta\geqslant 0$,*

$$r\alpha\,\pi(\alpha,\beta)-r\,\bar p(\beta)=(\varphi(0)+\beta)\,\pi(\alpha,\beta)-\frac{\varphi(0)-\varphi(\alpha)}{\alpha}-\varphi(\alpha)\,\pi(\alpha,\beta).$$

▷ *Model 2: identification of $\bar p(\beta)$.* Our next goal is to identify $\pi(\alpha,\beta)$, which requires us to find $\bar p(\beta)$. To this end, observe that, for any $\alpha,\beta\geqslant 0$,

$$\pi(\alpha,\beta)=\frac{r\,\bar p(\beta)-(\varphi(0)-\varphi(\alpha))/\alpha}{r\alpha-\varphi(0)+\varphi(\alpha)-\beta}.$$

Notice that $\varphi(\alpha)$ is the Laplace transform of a nonnegative function, and hence a convex decreasing function. It is easily seen that the denominator has exactly one positive real zero $\alpha^\star(\beta)$ for every positive real β, and this allows one to determine the one unknown quantity ¹V°. We call, for a given value of V ¡ 0, the unique positive root of the denominator U¢¹V°• As this should be a root of the numerator as well, we obtain

$$¹V° = \frac{1}{A} \frac{j¹0° \quad j¹U¢¹V°°}{U¢¹V°}•$$

We have proven the following result.

Theorem 9.1 *For any U- V ¡ 0,*

$$c¹U- V° = \frac{¹j¹0° \quad j¹U¢¹V°°°•U¢¹V° \quad ¹j¹0° \quad j¹U°°•U}{AU \quad j¹0° ¸ j¹U° \quad V}•$$

9.3 Interarrival time being correlated with previous claim size

This section defines a model in which a claim interarrival time is correlated with the previous claim size, and presents the analysis of the corresponding ruin probability, as usual through its transform. We use a formalism similar to the one introduced in Model 2 of Section 9.2; for technical reasons, however, the arrival rate can attain only finitely many values, as will become clear from the analysis.

B *Model and dependence structure.* We start by detailing the intended dependence structure. Consider a vector z ¹l₀-•••-l₃° such that 0 = l₀ Ÿ l₁ Ÿ Ÿ l₃ = 1. As before, let the claim sizes ₁- ₂-••• be a sequence of i.i.d. random variables; they are distributed as a generic, non-negative, random variable . In addition, +₁-+₂-••• is a second sequence of i.i.d. non-negative random variables, independent of the claim sizes; these are distributed as the generic random variable +. Regarding the claim interarrival times, the following mechanism applies. If a claim ₌ is in the interval »l₈ ₁+₌- l₈+₌°, then the time until the next claim is exponentially distributed with rate _₈ ¡ 0. As before, -D¹C° denotes the surplus process, with the initial reserve level being D.

The key object of our interest is the time-dependent ruin probability, which is in this context given by, for 8 = 1-•••-3,

$$?₈¹D-C° := P¹9B 2 ¹0-C¼ : -D¹B° Ÿ 0 j \quad ¹0° = 8°;$$

here the event f ¹0° = 8g corresponds to the scenario that the arrival rate at time 0 is _₈. The objective of this section is to characterize ?₈¹D-C° through its double transform

$$c₈¹U- V° = \int_0^1\int_0^1 V4^{UD \quad VC} ?₈¹D-C° \,dD\,dC-$$

for U > 0 and V ¡ 0 and 8 = 1-•••-3.

B *Transform of the ruin probability.* We analyze the quantity c₈¹U- V° by applying a hybrid method, combining elements from Method 1 (see Section 1.3) and Method 4 (see Section 1.6). Consider the possible events in the first C time units: this can be a claim arrival (only leading to ruin when the claim size exceeds D), killing, or none of these. We thus obtain, up to $>^1\ C^o$ terms, with $)_V$ as usual exponentially distributed with parameter V, that

$$?_8{}^1D\text{-})_V{}^o = \_{}_8\ C \sum_{9=1}^{3} \int_0^D P^1\ 2\,dE^o\,P + 2 \text{»} \frac{E}{I_9} \text{-} \frac{E}{I_{9\ 1}}{}^o\ ?_9{}^1D\ \ E\text{-})_V{}^o\ \text{¸}$$
$$\_{}_8\ C\,P^1\ > D^o\ \text{¸}\ 1\ \ \_{}_8\ C\ \ V\ C\ ?_8{}^1D\ \text{¸}\ A\ \ C\text{-})_V{}^o\text{-}$$

as C # 0. Subtracting $?_8{}^1D\ \text{¸}\ A\ \ C\text{-})_V{}^o$ from both sides and subsequently dividing the full equation by C, and then sending C to 0 leads to the following system of 3 integro-differential equations:

$$A\frac{m}{mD}?_8{}^1D\text{-})_V{}^o = \_{}_8 \sum_{9=1}^{3} \int_0^D P^1\ 2\,dE^o\,P + 2 \text{»} \frac{E}{I_9} \text{-} \frac{E}{I_{9\ 1}}{}^o\ ?_9{}^1D\ \ E\text{-})_V{}^o\ \text{¸}$$
$$\_{}_8\,P^1\ > D^o\ \ {}^1\_{}_8\ \text{¸}\ V^o\ ?_8{}^1D\text{-})_V{}^o\text{•} \qquad (9.5)$$

This system can be further analyzed by transforming it with respect to D. To this end we define, for $9 = 1\text{-}\cdots\text{-}3$,

$$j_9{}^1U^o := \int_0^1 4^{\ UE}\,P^1\ 2\,dE^o\,P + 2 \text{»} \frac{E}{I_9} \text{-} \frac{E}{I_{9\ 1}}{}^o\ \text{•}$$

The next step is to multiply Equation (9.5) by $4^{\ UD}$ and integrate over D. Applying the identity (1.9) to the left-hand side, and recognizing a convolution in the right-hand side, we obtain that Equation (9.5) can be rewritten in the following form, with $\ _8{}^1V^o := ?_8{}^1 0\text{¸-})_V{}^o$.

Proposition 9.2 *For any $U > 0$ and $V\ ¡\ 0$, and $8 = 1\text{-}\cdots\text{-}3$,*

$$AU\,c_8{}^1U\text{-}\,V^o\ \text{¸}\ A\ \ _8{}^1V^o = \_{}_8 \sum_{9=1}^{3} j_9{}^1U^o\,c_9{}^1U\text{-}\,V^o\ \text{¸}$$
$$\_{}_8\frac{1\ \ 1^1U^o}{U}\ \ {}^1\_{}_8\ \text{¸}\ V^o\,c_8{}^1U\text{-}\,V^o\text{•}$$

B *Solving the linear equations.* For a given value of $U > 0$ and $V\ ¡\ 0$, Proposition 9.2 provides a linear system of equations, containing the 3 unknowns $\ _1{}^1V^o\text{-}\cdots\text{-}\ _3{}^1V^o$. To identify these, we first observe that the equations immediately yield that, for any pair $8\text{-}\,9\ 2\ f1\text{-}\cdots\text{-}3g$,

$$\frac{AU\ \ \_{}_8\ \ V\ c_8{}^1U\text{-}\,V^o\ \ A\ \ _8{}^1V^o}{\_{}_8} = \frac{AU\ \ \_{}_9\ \ V\ c_9{}^1U\text{-}\,V^o\ \ A\ \ _9{}^1V^o}{\_{}_9}\text{-}$$

or, equivalently,

$$\phi_j(s,\delta) = a_{ij}(s,\delta)\,\phi_i(s,\delta) + b_{ij}(s,\delta), \tag{9.6}$$

where

$$a_{ij}(s,\delta) := \frac{\lambda_j}{\lambda_i}\frac{cs-\lambda_i-\delta}{cs-\lambda_j-\delta}, \qquad b_{ij}(s,\delta) := -\frac{\lambda_j}{\lambda_i}\frac{c\,\phi_i(\delta)}{cs-\lambda_j-\delta} + \frac{c\,\phi_j(\delta)}{cs-\lambda_j-\delta}.$$

Inserting (9.6) in the system of equations of Proposition 9.2, one readily obtains that, for all $i = 1,\dots,n$,

$$\begin{aligned}(-cs+\lambda_i+\delta)\,&\phi_i(s,\delta) + c\,\phi_i(\delta)\\ &= \lambda_i \sum_{j=1}^{n} \beta_j(s)\left(a_{ij}(s,\delta)\,\phi_i(s,\delta) + b_{ij}(s,\delta)\right) + \lambda_i \frac{1-w(s)}{s}.\end{aligned}$$

From this equation, the transform $\phi_i(s,\delta)$ can be solved, in terms of the unknowns $\phi_i(\delta)$. Indeed, for any $s > 0$ and $\delta \geq 0$,

$$\phi_i(s,\delta) = \frac{c\,\phi_i(\delta) - \lambda_i \sum_{j=1}^{n} \beta_j(s)\, b_{ij}(s,\delta) - \lambda_i (1-w(s))/s}{cs - \lambda_i + \lambda_i \sum_{j=1}^{n} \beta_j(s)\, a_{ij}(s,\delta) - \delta}. \tag{9.7}$$

Any zero of the denominator should be a zero of the numerator as well. Therefore equate, for $\delta \geq 0$ given and for $i = 1,\dots,n$, the denominator to 0, or, equivalently, $\Delta(s) = 1$, with

$$\Delta(s) = \sum_{j=1}^{n} \frac{\lambda_j}{\lambda_j + \delta - cs}\,\beta_j(s).$$

Observe that $\Delta(0) < 1$, whereas $\Delta(s)$ approaches 0 from below as $s \to \infty$. Let the λ_j be distinct (where we note that our analysis below immediately extends to the situation that some of the λ_j are equal), so that we can assume without loss of generality that $\lambda_1 < \dots < \lambda_n$. With $s_0 = 0$ and $s_j := (\lambda_j + \delta)/c$, we have that

$$\lim_{s \uparrow s_j} \Delta(s) = \infty, \qquad \lim_{s \downarrow s_j} \Delta(s) = -\infty,$$

for $j = 1,\dots,n$. We thus conclude that, for all $\delta \geq 0$, there is a solution to $\Delta(s) = 1$ in each of the intervals (s_{j-1}, s_j), for $j = 1,\dots,n$. We call these zeroes $s_1^*(\delta),\dots,s_n^*(\delta)$, which are then necessarily zeroes of the numerator as well. It therefore follows that, for $s = s_1^*(\delta),\dots,s_n^*(\delta)$,

$$\begin{aligned} N_i(s,\delta) &:= c\,\phi_i(\delta) - \lambda_i \sum_{j=1}^{n} \beta_j(s)\, b_{ij}(s,\delta) - \lambda_i \frac{1-w(s)}{s}\\ &= c\,\phi_i(\delta)\left(1 - \Delta(s)\right) + \lambda_i c \sum_{j=1}^{n} \frac{\beta_j(s)}{\lambda_j + \delta - cs}\,\phi_j(\delta) - \lambda_i \frac{1-w(s)}{s}\end{aligned}$$

should equal 0. Using that $(\alpha_j^{\star}(\beta)) = 1$ for $j = 1,\dots,d$, we obtain the following linear system, after dividing by λ_i:

$$\sum_{j=1}^{d} \frac{\lambda_j (\alpha_j^{\star}(\beta))}{\lambda_j + \beta - c\alpha_j^{\star}(\beta)} \, {}_j(\beta) = \frac{1 - (\alpha_j^{\star}(\beta))}{\alpha_j^{\star}(\beta)}, \tag{9.8}$$

for $j = 1,\dots,d$; observe that these equations do not depend on i anymore. From the linear equations displayed in (9.8) the probabilities ${}_j(\beta)$ can be solved in the usual manner. We have thus found the following result.

Theorem 9.2 *For any $\alpha > 0$ and $\beta \geq 0$, and $i = 1,\dots,d$, $c_i(\alpha,\beta)$ is given by* (9.7), *where the ${}_j(\beta)$, for $j = 1,\dots,d$, follow from the d linear equations* (9.8).

Remark 9.2 Notice that above we did not prove that the equation $(\alpha) = 1$ has *exactly d* real solutions, and that it has no non-real zeroes for Re $\alpha > 0$. If there were more than d zeroes, then the numerator of (9.7) would have to be zero for more α-values, leading to more than d linear equations in the d unknown ${}_j(\beta)$. This should result in superfluous equations. We discuss related issues in Remark 9.4. ^

Remark 9.3 If the arrival rates are equal, i.e., $\lambda_1 = \cdots = \lambda_d \equiv \lambda$, then we recover the known formula for $c(\alpha,\beta)$, as was given in Equation (1.4). Indeed, with as before $\kappa(\alpha) = c\alpha - \lambda(1 - (\alpha))$ and denoting by $\psi(\beta)$ its inverse, we find that ${}_{ij}(\alpha,\beta) \equiv 1$, so that the denominator of (9.7) becomes $c\alpha - \lambda(1 - (\alpha)) - \beta$ and hence $\alpha^{\star}(\beta) = \psi(\beta)$. We also have that ${}_i(\beta) = {}_j(\beta)$ for all $i,j \in \{1,\dots,d\}$, and therefore ${}_{ij}(\alpha,\beta) \equiv 0$. An elementary computation yields that $c_i(\alpha,\beta)$ equals the expression displayed in Equation (1.4). ^

9.4 A more general Markov-dependent risk model

In this section we consider a specific semi-Markovian dependence structure between claim interarrival times and claim sizes. The underlying model is, in line with the setup of the previous section, such that any interarrival time depends on the previous claim size.

Model and dependence structure. Let ${}_i$ denote the time between the arrival of the $(i-1)$-st and the i-th claim and ${}_0 = {}_0 = 0$. Then

$$\begin{aligned} &\mathbb{P}({}_{n+1} \leq x, {}_{n+1} \leq y, J_{n+1} = j \mid J_n = i, ({}_r, {}_r, J_r), r \in \{0,1,\dots,n\}) \\ &= \mathbb{P}({}_1 \leq x, {}_1 \leq y, J_1 = j \mid J_0 = i) = (1 - e^{-{}_i x})\, p_{ij}\, {}_j(y), \end{aligned} \tag{9.9}$$

where $(J_n)_{n \in \mathbb{N}}$ is an irreducible discrete-time Markov chain with finite state space $\{1,\dots,d\}$ and transition matrix P consisting of the transition probabilities $p_{ij} := \mathbb{P}(J_{n+1} = j \mid J_n = i)$. Thus at each instant of a claim, the Markov chain jumps to a state j, and the distribution function ${}_j(\cdot)$ of the claim size depends on the new state j. Then the next interarrival time is exponentially distributed with parameter λ_i. Observe that given the states Z_{n-1} and Z_n, the random quantities U_n and T_n are independent, but there is autocorrelation among consecutive claim sizes and among consecutive interclaim times as well as cross-correlation between U_n and T_n.

Our aim is to analyze the probability of ruin before T_β in this risk model. Notice that the model introduced above covers the models with causal dependence structures of the type that was considered in Section 9.3. To see this, choose a generic claim size U and for all $i = 1, \dots, d$, $p_{ij} = \mathbb{P}(U \in A_j)$ for some (possibly random) interval $A_j \subset \mathbb{R}$ and choose the distribution function $F_j(\cdot)$ to correspond to the conditional distribution of U given that $U \in A_j$. But the new setup is actually richer than the one of Section 9.3: if the state-dependent transition probability p_{ij} is defined as the probability that $U \in A_j$ conditional on the current state being i, then we arrive at a more general model that also allows the claim size itself to depend on the state of the Markov chain.

Transform of the ruin probability. Let $\psi_i(u, T_\beta)$ denote the ruin probability given that $Z_0 = i$. We shall derive a set of d integro-differential equations for these $\psi_i(u, T_\beta)$, relying on Method 4 of Chapter 1. Considering all possible transitions in an interval of length Δt, as $\Delta t \downarrow 0$, we derive in the standard manner

$$\psi_i(u, T_\beta) = \lambda_i \Delta t \sum_{j=1}^{d} p_{ij} \int_0^u \psi_j(u-y, T_\beta)\, F_j(\mathrm{d}y) + \lambda_i \Delta t \sum_{j=1}^{d} p_{ij} \bigl(1 - F_j(u)\bigr) + \bigl(1 - \lambda_i \Delta t - \beta \Delta t\bigr)\, \psi_i(u + r\Delta t, T_\beta) + o(\Delta t),$$

for $i = 1, \dots, d$. Rearranging and sending Δt to 0 yields

$$r \frac{\partial}{\partial u} \psi_i(u, T_\beta) = -\lambda_i \sum_{j=1}^{d} p_{ij} \int_0^u \psi_j(u-y, T_\beta)\, F_j(\mathrm{d}y) - \lambda_i \sum_{j=1}^{d} p_{ij} \bigl(1 - F_j(u)\bigr) + (\lambda_i + \beta)\, \psi_i(u, T_\beta), \tag{9.10}$$

for $i = 1, \dots, d$; cf. Equation (9.5). As in Section 9.3, our objective is to evaluate, for $\alpha > 0$ and $\beta \geq 0$,

$$\phi_i(\alpha, \beta) := \int_0^\infty e^{-\alpha u} \psi_i(u, T_\beta)\, \mathrm{d}u = \int_0^\infty \int_0^\infty \beta e^{-\alpha u - \beta t} \psi_i(u, t)\, \mathrm{d}u\, \mathrm{d}t,$$

expressed in terms of quantities $\varphi_i(\alpha)$ which are now defined by

$$\varphi_i(\alpha) := \int_0^\infty e^{-\alpha y} F_i(\mathrm{d}y).$$

This is done following the procedure that has become standard by now, i.e., multiplying Equation (9.10) by $e^{-\alpha u}$ and integrating over u. After a straightforward calculation, we obtain the following result, which can be considered as the counterpart of Proposition 9.3. As in Section 9.3, we use the notation $_8{}^1V^o := ?_8{}^1 0 ¸ -)_V{}^o•$

Proposition 9.3 *For any* $U > 0$ *and* V ¡ 0*, and* $8 = 1 - • • • - 3$*,*

$$AU\, c_8{}^1U - V^o \; ¸ \; A \quad _8{}^1V^o = \_8 \sum_{9=1}^{3} ?_{89}\; j_9{}^1U^o\, c_9{}^1U - V^o \; ¸$$

$$\_8 \sum_{9=1}^{3} ?_{89} \frac{1 \quad j_9{}^1U^o}{U} \quad {}^1\_8 \; ¸ \; V^o\, c_8{}^1U - V^o•$$

B *Solving the linear equations.* We start by writing the system of equations of Proposition 9.3 in a convenient matrix form: in self-evident notation,

$$^1AU \quad V^o \qquad ¸ \quad \% - {}^1U^o \;\; \pi^1U - V^o = A f^1V^o \qquad \% \frac{1}{U}{}^1 \qquad - {}^1U^{oo} - \qquad (9.11)$$

where $:= \mathrm{diag}^1\lambda^o$ and $- {}^1U^o := \mathrm{diag}^1\chi^1U^{oo}$. When solving this system of linear equations in the 3 functions $c_8{}^1U - V^o$, it remains to determine the 3 unknown constants $_8{}^1V^o$. This is done by studying the zeroes of

$$_V{}^1U^o := {}^1AU \quad V^o \qquad ¸ \quad \% - {}^1U^o•$$

The following lemma is useful in this context.

Lemma 9.1 *For any* V ¡ 0*, the equation* $\det \; _V{}^1U^o = 0$ *has* 3 *zeroes* $U_1 - • • • - U_3$ *with positive real part.*

Proof: Let denote a circle with its center at

$$2_0 := \frac{V \; ¸ \; \max_{82f1-•••-3g} \_8}{A} -$$

and radius 2_0. Also define for $E\ 2$ »0- 1¼ the matrix

$$_V{}^1U - E^o := {}^1AU \quad V^o \qquad ¸ \; E \quad \% - {}^1U^o;$$

observe that $_V{}^1U - 1^o = \; _V{}^1U^o$. Our first goal is to prove, for $E\ 2$ »0- 1¼, that

$$\det \; _V{}^1U - E^o < 0 \text{ for } U\ 2 \quad • \qquad (9.12)$$

It is readily verified that the matrix $_V{}^1U - E^o$ is diagonally dominant for any $E\ 2$ »0- 1¼. Indeed,

$$\begin{aligned} jAU \quad V \quad \_8 \; ¸ \; E\_8\, ?_{88}\; j_8{}^1U^o j &> jV \; ¸ \; \_8 \quad AUj \quad jE\_8\, ?_{88}\; j_8{}^1U^o j \\ &> V \; ¸ \; \_8 \quad E\_8\, ?_{88}\; j_8{}^1 0^o \\ &¡ \; E\_8{}^1 1 \quad ?_{88}\; j_8{}^1 0^{oo} \end{aligned}$$

$$= E\_8 \sum_{9<8} ?_{89}\, j_9{}^1 0^o > E\_8 \sum_{9<8} ?_{89}\, j_9{}^1 U^o \bullet$$

It is a standard matrix-theoretic result that this diagonal dominance [10, pp. 146-147] directly implies (9.12).

Now let $=^1E^o$ denote the number of zeroes of det $_V{}^1U\text{-}E^o$ in , the interior of . Then, cf. [8, p. 97],

$$=^1E^o = \frac{1}{2ci}\int^1 \frac{{}^{0\,1}U\text{-}E^o}{{}^1U\text{-}E^o}\,\mathrm{d}U\text{-}$$

where

$$^1U\text{-}E^o := \det \quad _V{}^1U\text{-}E^o\text{-} \qquad {}^{0\,1}U\text{-}E^o := \frac{m}{mU} \quad {}^1U\text{-}E^o \bullet$$

Observe that $=^1E^o$ is a continuous function on »0- 1¼, integer-valued, and therefore constant. Then note that $=^10^o = 3$, because det $_V{}^1U\text{-}0^o = \det^{11}AU \quad V^o \qquad {}^o = 0$ for $U = {}^1V \,_{\text{¸}}\, \_{8}{}^o \bullet A$, with $8 = 1\text{-}\bullet\bullet\bullet\text{-}3$. Conclude that also $=^11^o = 3$.

Remark 9.4 It follows from the above proof that the zeroes are located in the interior . It should be observed that, formally, we have only shown that det $_V{}^1U^o$ has 3 zeroes in , leaving open the possibility that there are additional zeroes outside in the right-half U-plane. However, we only need 3 linear equations to determine the 3 unknown constants $_8{}^1V^o$, and we also know that those constants exist. So if there were more zeroes in the right-half U-plane, then this should result in superfluous equations; see also Remark 9.2. ^

Remark 9.5 From [1, Theorem 3.2] it directly follows that for $V = 0$ the equation det $_0{}^1U^o = 0$ has one zero $U_1 = 0$ as well as $3 \quad 1$ zeroes $U_2\text{-}\bullet\bullet\bullet\text{-}U_3$ with positive real part. If the claim size distributions have a rational LST, then the ruin probability can be obtained explicitly by inversion of the Laplace transform of the solution of (9.11). ^

With Lemma 9.1 at our disposal, we can proceed along the following lines. Observe that $c_8{}^1U\text{-}V^o$ are analytic in U with positive real part, so that for each of the 3 zeroes $U_1\text{-}\bullet\bullet\bullet\text{-}U_3$ we can apply the following procedure. Determine, for each $8 = 1\text{-}\bullet\bullet\bullet\text{-}3$, a non-trivial solution $\boldsymbol{g}_8$ of the linear system

$$\overset{>}{_V}{}^1U_8{}^o\, \boldsymbol{g}_8 = \boldsymbol{0}\bullet$$

Since we then have

$$0 = \boldsymbol{\pi}^1U_8\text{-}\,V^{o>} \quad \overset{>}{_V}{}^1U_8{}^o\, \boldsymbol{g}_8 = {}^1A\boldsymbol{f}^1V^o \qquad \% \frac{1}{U_8}{}^1 \qquad \text{-}\,{}^1U_8{}^{oo>}\, \boldsymbol{g}_8\text{-} \tag{9.13}$$

this provides us with 3 linear equations for $5_1{}^1V^o\text{-}\bullet\bullet\bullet\text{-}\,5_3{}^1V^o$. The following result summarizes our findings.

Theorem 9.3 *For any $U > 0$ and V ¡ 0,*

$$\pi(U,V) = (\quad V(U))^{1} \; A f(V) \qquad \% \frac{1}{U}(\qquad ,(U)) ,$$

where the $_{9}(V)$*, for* $9 = 1,\ldots,3$*, follow from the 3 linear equations* (9.13).

9.5 Discussion and bibliographical notes

An early study of an insurance risk model in which, as in Section 9.2, a claim size depends on the preceding interclaim time, is [4]; they consider dependence according to an arbitrary copula structure.

The model discussed in Section 9.3 is based on, but generalizes, the results presented in [2]. Notably, [2] does not consider the finite (exponentially distributed) horizon, and focuses on the case $3 = 2$ only. We refer to [6] for the analysis of a related queueing model. A discussion of existence/uniqueness issues is given in [7], in line with the observations that we made in Remarks 9.2 and 9.4, but then in a related queueing setting.

Section 9.4, featuring a semi-Markovian dependence structure, is based on [3]. The proof of Lemma 9.1 follows an idea in [11] (see also [1]); we have repeated the line of reasoning of [11] essentially verbatim, as it has the potential to be applied more broadly in the context of insurance risk. Adan and Kulkarni [1] consider a similar semi-Markov dependence as in Section 9.4, but for a queueing model and with $\__{8}$ replaced by $\__{9}$ in (9.9). The model in Section 9.4 belongs to a more general class of risk models where claims arrive according to a *Markovian Arrival Process* (MAP). MAP risk models can be studied via a connection to fluid queues when the claim amounts are of phase-type; see, e.g., the survey [5].

Another generalization concerning the arrival process, is the extension to *renewal* claim arrivals: where in the conventional Cramér-Lundberg model the claim interarrival times are i.i.d. exponentially distributed random variables, in the Sparre Andersen model the claim interarrival times are i.i.d. samples from some general distribution on the positive halfline. In [12] an extensive account of ruin theory for renewal claim arrivals is provided.

Exercises

9.1 As pointed out in Section 1.7, one is often not interested in just the ruin probability, but in various related measures as well, such as the ruin time $g(D)$ (on the event that ruin occurs), the undershoot $-_{D}(g(D) \;)$, and the overshoot $-_{D}(g(D))$. In this exercise we determine these 'Gerber-Shiu metrics' [9] for the model of Section 9.4, thus reproving results from [3]. In this exercise we adapt Proposition 9.3 so as to cover the case that $c_{8}(U,V)$ is replaced by $c_{8}(U,V,\gamma)$, defined analogously to the transform $c(U,V,\gamma)$ that was introduced in Exercise 1.2 for the conventional Cramér-Lundberg model. Define

$$?_8{}^1D\text{-}C\text{-}\gamma^o := E\ 4^{\ W_1 g^1 D^o\ \ W_2 - D^1 g^1 D^o\ \ ^o\ \ W_3 - D^1 g^1 D^{oo}} 1 f g^1 D^o\ 6\ C g j\ \ ^1 0^o = 8\ \bullet$$

In the sequel we abbreviate $?_8{}^1D\text{-})_V\text{-}\gamma^o$ to just $?_8{}^1D^o$.

(i) Show that, as $C \# 0$,

$$?_8{}^1D^o = 4^{\ W_1\ C}\ \_{}_8\ C \quad ?_{89} \quad ?_9{}^1D \quad E^o \quad {}_9{}^1\mathrm{d}E^o\ ,$$

$$\_{}_8\ C \quad ?_{89} \quad 4^{\ W_2 D\ \ W_3{}^1 D\ \ E^o}\ \ {}_9{}^1\mathrm{d}E^o\ ,$$

$$1 \quad \_{}_8\ C \quad V\ C\ ?_8{}^1D\ ,\ A\ \ C^o\ \text{-}$$

for $8 = 1\text{-}\bullet\bullet\bullet\text{-}3$, up to $>^1\ C^o$-terms.

(ii) By following the same procedure as in Exercise 1.2, derive the counterpart of Equation (9.10).

(iii) By transforming the result of (ii) with respect to D, derive the counterpart of Proposition 9.3.

9.2 Consider the following two variants of the model of Section 9.3, with $3 = 2$ and with $)_8 = I_1 +_8$ the ‘threshold’.

(i) In the first variant, the claim arrival rate only changes when ${}_8\ \ddot{Y}\ )_8$. Investigate the question how Proposition 9.2 and Theorem 9.2 can be adapted.

(ii) Same question for the second variant, in which we have the following two scenarios. If ${}_8\ \ddot{Y}\ )_8$, then the next claim arrival rate is $\_{}_1$ and the claim size is ${}_8$. If ${}_8 >)_8$ then the next claim arrival rate is $\_{}_2$ and the real claim size is $)_8$; so the real claim size always is $\min f\ {}_8\text{-})_8 g$. The threshold $)_8$ can thus be interpreted as the retention level of an XL-type reinsurance of the claim size, i.e., the amount an insurer assumes for its own account in the case of excess-of-loss reinsurance.

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Chapter 10
Advanced bankruptcy concepts

Abstract — So far we focused on the event of *ruin*, corresponding to the reserve process dropping below 0. This chapter studies various *bankruptcy* concepts, in which, besides the reserve level becoming negative, an additional condition has to be fulfilled. In the first variant, the net reserve process is inspected at Poisson instants, and bankruptcy occurs if the reserve level is below zero at such an inspection time. For this setting an adapted version of the Pollaczek-Khinchine theorem is derived, as well as an appealing decomposition. In the second variant there is bankruptcy if the reserve process is *uninterruptedly* below 0 for a sufficiently long time, whereas in the third variant the bankruptcy criterion is based on the *total* time with a negative surplus.

10.1 Introduction

The focus of the previous chapters was on the evaluation of *ruin* probabilities, assessing the likelihood of the event that the insurance firm's surplus level drops below 0 (typically before an exponentially distributed clock expires). This chapter considers various more sophisticated bankruptcy concepts, in which, besides a negative surplus level, a second criterion needs to be fulfilled.

Three variants are considered in this chapter, each of them with an own definition of bankruptcy.

In the first variant, the insurance firm's reserve process cannot be continuously monitored, but is rather only inspected at Poisson times. Bankruptcy occurs when at such an inspection time the surplus level is below 0. We evaluate the LST of the running maximum of the net cumulative claim process (at inspection times, that is), which uniquely defines the corresponding bankruptcy probability. This LST is a generalization of (the time-dependent version of) the Pollaczek-Khinchine formula; indeed, as the inspection rate increases, the expression converges to the known formula (corresponding to continuous inspection). Our second main contribution is an appealing decomposition that relates the running maximum of the net cumulative claim process at inspection moments to its continuously inspected counterpart. In the proofs, a connection with the M/G/1 queue is exploited.

In the second variant, the criterion is that when the surplus level (first) drops below level 0, bankruptcy occurs when the time spent below 0 is sufficiently long. To be able to assess the probability of bankruptcy, we derive the transform of the length of the first excursion of the surplus process below level 0 (up to an exponentially distributed killing time, that is), relying on various results that were obtained in previous chapters.

In the third variant bankruptcy occurs when the aggregate time the surplus process spends below level 0 is larger than some threshold. To quantify the bankruptcy probability, we analyze the transform of the total time that the insurance firm faces a negative surplus, up to an exponentially distributed killing time.

This chapter is organized as follows. In Section 10.2 we consider the case of Poisson inspection times, leading to the LST of the running maximum as well as the decomposition result. In addition, corresponding tail asymptotics are given, shedding light on the difference between the ruin probability and the bankruptcy probability. Section 10.3 analyzes the length of the reserve process' first excursion below level 0, and Section 10.4 considers the total time spent below 0.

10.2 Poisson inspection times

As before, we consider the probability of the insurance firm's surplus level dropping below 0, but now with the complication that the surplus level is only observed at the Poissonian inspection epochs $\tau_1, \tau_2, \ldots$, i.e., *not* continuously in time. The Poisson inspection times entail that the times between two subsequent inspections (i.e., $\tau_n - \tau_{n-1}$ for $n \in \mathbb{N}$, with $\tau_0 \equiv 0$) are i.i.d. exponentially distributed random variables, say with parameter $\omega > 0$. The quantity we wish to analyze is

$$p(u,t) := \mathbb{P}(\exists n \in \mathbb{N} : \tau_n \leqslant t, X_u(\tau_n) < 0),$$

which we will refer to as the (time-dependent) *bankruptcy probability* given that the initial reserve level is u. It is clear that $p(u,t) \leqslant \bar{p}(u,t)$, as in case no bankruptcy occurs the reserve level could still attain non-positive values in between inspection epochs; see Figure 10.1.

To analyze the bankruptcy probability $p(u,t)$ we consider the net cumulative claim process $Y(t)$ at the Poisson inspection epochs $\tau_1, \tau_2, \ldots$. In addition, we impose 'killing', in the sense that we focus on bankruptcy taking place within a horizon that is exponentially distributed with parameter $\vartheta > 0$. Concretely, we work with the increments of $Y(t)$ between two consecutive inspections, i.e.,

$$B_n := Y(\tau_n) - Y(\tau_{n-1}).$$

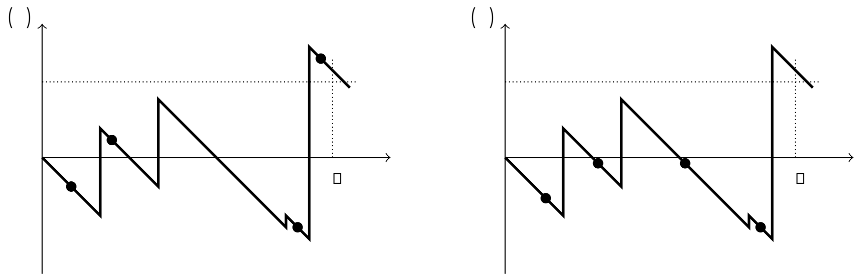

Fig. 10.1 A scenario with ruin and bankruptcy before the killing time T_β (left panel), and a scenario with ruin but no bankruptcy before T_β (right panel). The black dots indicate the Poisson inspection epochs.

which form a sequence of i.i.d. random variables, distributed as the generic random variable Y. The killing time T_β is, as before, an exponentially distributed random variable with parameter β, independent of the surplus process. It is easily seen that the number of inspections before killing, denoted by $N \equiv N_{\beta,\omega}$, has a geometric distribution with success probability $\beta/(\beta+\omega)$:

$$\mathbb{P}(N=n) = \left(\frac{\omega}{\beta+\omega}\right)^n \frac{\beta}{\beta+\omega}, \quad n = 0,1,\ldots.$$

Now define the following running maximum process:

$$\bar{Y}_{\beta,\omega} := \sup_{n=0,1,\ldots,N_{\beta,\omega}} Y(T_n) = \sup_{n=0,1,\ldots,N_{\beta,\omega}} \sum_{i=1}^{n} Y_i; \tag{10.1}$$

here and in the sequel a maximum over an empty set is defined to be zero. Notice that we can rewrite our bankruptcy probability as

$$p(u,T_\beta) = \mathbb{P}(\bar{Y}_{\beta,\omega} > u).$$

In the remainder of this section we analyze $p(u,T_\beta)$ by evaluating $\mathbb{P}(\bar{Y}_{\beta,\omega} > u)$. In our analysis an important role is played by a characterization of the transient waiting time of the M/G/1 queue. As an intermezzo, we now derive the transform of this transient waiting time, a result that we need in our analysis.

▷ *Transient waiting time in the M/G/1 queue.* We point out how to compute the transform of the waiting time W_N of the N-th customer, with N being geometrically distributed with success probability $q \in (0,1)$. As mentioned, this will be used in our analysis of the transform of the bankruptcy probability $p(u,T_\beta)$. Let in our M/G/1 system, which we assume to start empty, the arrival rate be $\lambda > 0$, and let the non-negative jobs be given by the sequence of i.i.d. random variables $(B_n)_{n\in\mathbb{N}}$ that are distributed as a generic random variable B with LST $b(\alpha) := \mathbb{E}\, e^{-\alpha B}$.

The starting point of the derivation of the transform of W_N is the so-called *Lindley recursion*: it is readily verified that W_{n+1} can be expressed in terms of W_n through the relation

$$W_{n+1} = \max\{W_n + B_n - T_{n+1}, 0\},$$

where $(T_n)_{n\in\mathbb{N}}$ is a sequence of independent exponentially distributed random variables with parameter λ, and where we assumed $W_0 = 0$. This leads to the identity, with $\varphi_n(\alpha) := \mathbb{E}\, e^{-\alpha W_n}$,

$$\varphi_{n+1}(\alpha) = \int_0^\infty \int_0^\infty e^{-\alpha \max\{x-y,0\}} \lambda e^{-\lambda y} \mathrm{d}y\, \mathbb{P}(W_n + B_n \in \mathrm{d}x).$$

Distinguishing between the cases $x \leqslant y$ and $x > y$, we find, after having evaluated the integral over y, that the expression in the previous display equals

$$\frac{\lambda}{\alpha-\lambda}\int_0^\infty (e^{-\lambda x} - e^{-\alpha x})\, \mathbb{P}(W_n + B_n \in \mathrm{d}x) + \int_0^\infty e^{-\lambda x}\, \mathbb{P}(W_n + B_n \in \mathrm{d}x),$$

which, using the independence of W_n and B_n, eventually leads to

$$\varphi_{n+1}(\alpha) = \frac{\alpha\, \varphi_n(\lambda)\beta(\lambda) - \lambda\, \varphi_n(\alpha)\beta(\alpha)}{\alpha - \lambda}.$$

We proceed by finding an expression for the LST of the waiting time of the N-th customer. To this end, multiplying both sides by $(1-q)^n q$ and summing over $n = 0, 1, \ldots$, yields

$$\mathbb{E}\, e^{-\alpha W_N} = \frac{q(\alpha - \lambda) + \alpha(1-q)\beta(\lambda)\,\mathbb{E}\, e^{-\lambda W_N}}{\alpha - \lambda + \lambda(1-q)\beta(\alpha)}. \tag{10.2}$$

We now determine the constant $\mathbb{E}\, e^{-\lambda W_N}$. Using Rouché's theorem (see Theorem A.1, and for this case in particular [13, p. 250]) it follows that the denominator in the righthand side of (10.2) has exactly one zero, α_0, with a positive real part. This root α_0 is actually real and can be written in a convenient form, as we now show. With

$$\psi(\alpha) := \mathbb{E}\, e^{-\alpha(B_n - T_{n+1})} = \frac{\lambda}{\lambda - \alpha}\beta(\alpha),$$

we are to solve $\psi(\alpha) = 1/(1-q)$. Defining $\psi^{-1}(\cdot)$ as the (right-)inverse of $\psi(\cdot)$, we have

$$\alpha_0 = \psi^{-1}\left(\frac{1}{1-q}\right).$$

Observing that $\psi(0) = 1$, that $\psi(\lambda) = \infty$, and that the function $\psi(\cdot)$ is convex on the interval $(-\infty, \lambda)$, in combination with the fact that $1/(1-q) > 1$, we conclude that the unique root α_0 lies between 0 and λ. Since $\mathbb{E}\, e^{-\alpha W_N}$ is analytic for $\alpha = \alpha_0$, the numerator of the right-hand side of (10.2) should be zero for $\alpha = \alpha_0$. We thus find that, with $\alpha_0 = \psi^{-1}((1-q)^{-1})$,

$$\mathrm{E}4^{\;a\,,\,\#} = \frac{@}{1\quad @}\,\frac{a\quad U_0}{U_0}\,\frac{1}{X^1a^o}\bullet$$

We have thus arrived at the following result, which can be seen as a counterpart of Theorem 1.1.

Lemma 10.1 *For* U ¡ 0 *and* $@$ 2 »0- 1¼,

$$\mathrm{E}4^{\;U\,,\,\#} = @\,\frac{U \quad a\ \text{¸}\ {}^1a \quad U_0{}^o\,U\bullet U_0}{U \quad a\ \text{¸}\ a^1 1 \quad @^o\,X^1U^o} = \frac{U}{U_0} \quad 1 \quad \frac{@a}{U \quad a\ \text{¸}\ a^1 1 \quad @^o\,X^1U^o}\bullet$$

The next lemma relates the customers' waiting times to an associated running maximum process. For convenience, denote $_= :=\ \ _{=\ 1}\ \ _=\bullet$

Lemma 10.2 *For any* = = 0- 1- • • • -

$$,_= \overset{\text{d}}{=} \max_{<=0\text{-}1\text{-}\bullet\bullet\bullet\text{-}=} \ \tilde{\Theta}_{8=1}\ \ _8 =:\ \ _=\bullet$$

Proof. By iterating the Lindley recursion,

$$\begin{aligned} ,_= &= \max f\,,_{=\ 1}\ \text{¸}\ \ _=\text{-}\,0g = \max f \max f\,,_{=\ 2}\ \text{¸}\ \ _{=\ 1}\text{-}\,0g\ \text{¸}\ \ _=\text{-}\,0g \\ &= \max f\,,_{=\ 2}\ \text{¸}\ \ _{=\ 1}\ \text{¸}\ \ _=\text{-}\ \ _=\text{-}\,0g\bullet \end{aligned}$$

Proceeding along these lines, after = iterations we arrive at the following representation of $,_=$ in terms of partial sums of the $_=$:

$$,_= = \max\left(\max_{<=1\text{-}\bullet\bullet\bullet\text{-}=} \ \tilde{\Theta}_{8=<}\ \ _8\text{-}\,0\right)\bullet$$

The stated follows by reversing time and recalling the maximum over an empty set was defined as 0.

By combining the above two lemmas, we find an expression for the transform of the running maximum process $^1\ \ _=^o{}_{=2\mathbb{N}}$, with the number of terms having a geometric distribution: we conclude that

$$\mathrm{E}4^{\;U\ \ \#} = @\,\frac{U \quad a\ \text{¸}\ {}^1a \quad U_0{}^o\,U\bullet U_0}{U \quad a\ \text{¸}\ a^1 1 \quad @^o\,X^1U^o} = \frac{U}{U_0} \quad 1 \quad \frac{@a}{U \quad a\ \text{¸}\ a^1 1 \quad @^o\,X^1U^o}\bullet \tag{10.3}$$

B *Evaluation of transform of running maximum at inspection epochs.* We use the above M/G/1 results to analyze the bankruptcy probability $?^1D\text{-}\,)_V{}^o$ by considering its associated double transform. Concretely, for U > 0 and V ¡ 0, we evaluate the transform

$$\bar{r}_{\,I}\,{}^1U\text{-}\,V^o := \mathrm{E}4^{\;U\,\bar{.}_{V\text{-}\,I}}\bullet$$

Recall that, by (10.1), $\bar{.}_{V\text{-}\,I}$ is the running maximum of the partial sums of the process $^1/_=^o{}_{=2\mathbb{N}}$, over maximally $\#_{V\text{-}\,I}$ terms.

Appealing to the Wiener-Hopf decomposition of Proposition 1.2, a key observation is that we can decompose the increments as

$$/ = /› \quad / \;-$$

with /› and / both non-negative and independent; see Figure 10.2. We proceed by considering the random variables / and /› in greater detail.

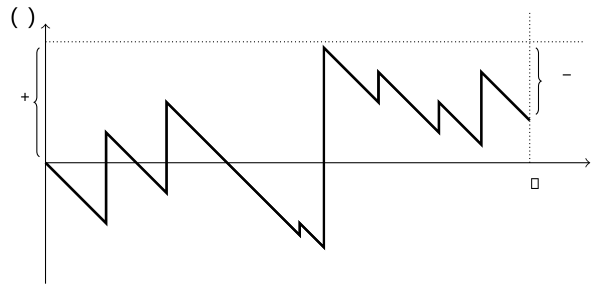

Fig. 10.2 The (Wiener-Hopf based) decomposition / = /› / , with /› and / non-negative and independent.

In the first place, observe that / is distributed, again by Proposition 1.2, as the running minimum of . ¹C° over a period that is exponentially distributed with parameter V ¸ I (to see this, recall that, conditional on)V i) I ,) I is exponentially distributed with parameter V ¸ I). We have seen in Section 1.3 that this running minimum has an exponential distribution with parameter k¹V ¸ I°; here k¹ ° is the right inverse of q¹U° = AU _¹1 1¹U°°.
The results of Section 1.3 also entail that

$$\mathrm{E}4^{\,U/›} = \frac{U \quad k^1V \,¸\, I^{o}}{i^1U^{o} \quad V \quad I}\,\frac{V \,¸\, I}{k^1V \,¸\, I^{o}}\,•$$

The consequence of the above observations is that $\bar{.}_{V- I}$ can be interpreted as the waiting time of the $\#_{V- I}$-th customer in an M/G/1 queue (with service speed 1), where $\#_{V- I}$ is geometrically distributed with success probability @ := V•¹V ¸ I°, the arrival rate is a := k¹V ¸ I°, and the jump sizes are distributed as the /¸, i.e.,

$$X^1U^{o} := \frac{U \quad k^1V \,¸\, I^{o}}{i^1U^{o} \quad V \quad I}\,\frac{V \,¸\, I}{k^1V \,¸\, I^{o}}\,•$$

Upon combining the above, we obtain from (10.3) that the transform of our interest can be written as

$$\bar\kappa_\omega(\alpha,\vartheta) = \frac{\vartheta}{\vartheta+\omega}\,\frac{\alpha-\Psi(\vartheta+\omega)+(\Psi(\vartheta+\omega)-\alpha_0)\frac{\alpha}{\alpha_0}}{\alpha-\Psi(\vartheta+\omega)+\Psi(\vartheta+\omega)\frac{\omega}{\vartheta+\omega}\frac{\alpha-\Psi(\vartheta+\omega)}{\varphi(\alpha)-\vartheta-\omega}\frac{\vartheta+\omega}{\Psi(\vartheta+\omega)}}.$$

It takes some elementary calculus to verify that this expression can be simplified to

$$\frac{\vartheta}{\vartheta+\omega}\,\frac{\Psi(\vartheta+\omega)}{\alpha-\Psi(\vartheta+\omega)}\left(\frac{\alpha}{\alpha_0}-1\right)\frac{\varphi(\alpha)-\vartheta-\omega}{\varphi(\alpha)-\vartheta}.$$

The next step is to identify α_0, which is a zero of the denominator of the expression in the previous display. Observe that one root of the denominator, viz. $\alpha=\Psi(\vartheta+\omega)$, is an obvious root of the numerator as well, so that we end up with the other root of the numerator, viz. $\alpha_0=\Psi(\vartheta)$ (which can readily be verified to be in agreement with our earlier statement that $\alpha_0 = ((1 - @)^{-1})$). After rearranging the factors in the numerators and denominators, we arrive at the following result.

Theorem 10.1 *For any $\alpha>0$ and $\vartheta\geqslant 0$,*

$$\bar\kappa_\omega(\alpha,\vartheta) = \frac{\alpha-\Psi(\vartheta)}{\varphi(\alpha)-\vartheta}\,\frac{\vartheta}{\Psi(\vartheta)}\,\frac{\varphi(\alpha)-\vartheta-\omega}{\alpha-\Psi(\vartheta+\omega)}\,\frac{\Psi(\vartheta+\omega)}{\vartheta+\omega}.$$

Observe that Theorem 10.1 is a true generalization of the time-dependent version of the Pollaczek-Khinchine formula, as was presented in Theorem 1.1. Indeed, as $\omega\to\infty$, which corresponds to the limiting regime of 'continuous inspection' of the process $X(t)$, we find that the transform $\bar\kappa_\omega(\alpha,\vartheta)$ coincides with the one appearing in the time-dependent Pollaczek-Khinchine formula. With $\kappa(\alpha,\vartheta)$ denoting the transform of $\bar X(T_\vartheta)$, as given in Theorem 1.1, we in addition find the appealing relation

$$\bar\kappa_\omega(\alpha,\vartheta) = \frac{\kappa(\alpha,\vartheta)}{\kappa(\alpha,\vartheta+\omega)}.$$

We also observe that $\bar\kappa_\omega(\alpha,\vartheta)=1$ as $\omega\downarrow 0$, as expected. The above equality, obviously being equivalent to $\kappa(\alpha,\vartheta)=\bar\kappa_\omega(\alpha,\vartheta)\,\kappa(\alpha,\vartheta+\omega)$, reveals the following remarkable distributional equality.

Theorem 10.2 *For any $\vartheta,\omega\geqslant 0$,*

$$\bar X(T_\vartheta) \stackrel{d}{=} \bar X(T_{\vartheta+\omega}) + \bar X_{\vartheta,\omega},$$

with the two random variables on the right-hand side being independently sampled.

Observe that $\bar X(T_{\vartheta+\omega})$ is decreasing in ω (as a supremum over an increasingly small interval is taken), whereas $\bar X_{\vartheta,\omega}$ is increasing in ω (as the inspection process takes place at an increasingly high frequency, with the length of the interval held fixed), but apparently these effects are equally strong, as evidenced by the fact that the left-hand side does not depend on ω.

The above decomposition has various applications. It can, for instance, be used to develop a recursive scheme to identify the moments of $\bar X_{\vartheta,\omega}$ from the moments of $\bar\tau(\sigma_\alpha)$ and $\bar\tau(\sigma_{\alpha+\omega})$; here it is noted that a recursive procedure to evaluate the moments of $\bar\tau(\sigma_\alpha)$ and $\bar\tau(\sigma_{\alpha+\omega})$ can be set up in a straightforward manner, relying on the generalized Pollaczek-Khinchine formula of Theorem 1.1 (see Exercise 1.1). Indeed, as a straightforward application of the binomium, exploiting the fact that the two terms in the right-hand side of the decomposition in Theorem 10.2 are independent, the following recursion can be derived. Informally, it quantifies the amount by which the Poisson-inspected system underestimates the continuously inspected system.

Proposition 10.1 *For any $k\in\mathbb{N}$,*

$$\mathbb{E}[(\bar\tau_{\alpha,\omega})^k] = \mathbb{E}[(\bar\tau(\sigma_\alpha))^k] - \sum_{\ell=1}^{k}\binom{k}{\ell}\,\mathbb{E}[(\bar\tau(\sigma_{\alpha+\omega}))^\ell]\,\mathbb{E}[(\bar\tau_{\alpha,\omega})^{k-\ell}].$$

As an example, we compute the first moment of $\bar\tau_{\alpha,\omega}$. This can be done relying on Proposition 10.1, but it is more straightforward to use Theorem 10.2:

$$\mathbb{E}\,\bar\tau_{\alpha,\omega} = \mathbb{E}\,\bar\tau(\sigma_\alpha) - \mathbb{E}\,\bar\tau(\sigma_{\alpha+\omega}).$$

In Remark 1.4, an expression for $\mathbb{E}\,\bar\tau(\sigma_\alpha)$ was given. We thus find

$$\mathbb{E}\,\bar\tau_{\alpha,\omega} = \frac{1}{\alpha}\varphi'(0) - \frac{1}{\alpha+\omega}\varphi'(0) - \frac{1}{\psi(\alpha)} + \frac{1}{\psi(\alpha+\omega)}.$$

When comparing this to $\mathbb{E}\,\bar\tau(\sigma_\alpha)$, we obtain insight into the underestimation due to Poisson inspections. We refer to Exercise 10.2, where it must be shown that $\mathbb{E}\,\bar\tau(\sigma_\alpha) - \mathbb{E}\,\bar\tau_{\alpha,\omega}$ behaves as $O(1/\omega)$ as $\omega\to\infty$, i.e., the underestimation vanishes as ω^{-1} in the regime that the inspection rate grows large.

▷ *Assessment of information loss due to Poisson inspection.* In Chapter 2 we have derived an approximation for $p(u)$ in the regime that u grows large. In particular, in the case of light-tailed input, we found positive constants γ and $\alpha^\star$ such that $p(u)\,e^{\alpha^\star u}\to\gamma$. Note that we have tacitly assumed that $X(t)$ does not tend to ∞, enforced by $\varrho<1$, thus ruling out $p(u)=1$ for all $u>0$.

One could wonder how much lower $\bar p(u) := p(u,\infty)$ is than $p(u)$; this would quantify the information loss due to the Poisson inspection mechanism (relative to continuous inspection, that is). As $\omega\to\infty$, we should have that $\bar p(u)$ tends to $p(u)$, but one may wonder whether we can quantify the speed of this convergence.

To study this effect, we recall a property that we have found earlier in this section: $\bar p(u)$ can be interpreted as the probability that an alternative net cumulative claim process $\bar X(t)$ (i.e., different from our actual net cumulative claim process $X(t)$) exceeds u. As we saw, this process $\bar X(\cdot)$ is defined by an arrival rate $\bar\lambda := \psi(\omega)$, claim sizes that are characterized by the LST

$$\bar b(s) := \frac{s-\psi(\omega)}{\varphi(s)-\omega}\,\frac{\omega}{\psi(\omega)},$$

and premium rate $A = 1$. The theory of Section 2.2 thus entails that there are constants $\mathbb{W}$ and $\bar{\ }^{¢}$ such that $?¹D^o 4^{\bar{\ }¢D}$! $\mathbb{W}$. Our goal is to compare the resulting asymptotics to those pertaining to $?¹D^o$•

Let us first consider W and $\backslash^{¢}$. As we discussed in Section 2.2, $\backslash^{¢}$ is the unique solution of the equation $i¹ \ \backslash^{¢o} = A\backslash^{¢} \ \_¹1 \ 1¹ \ \backslash^{¢oo} = 0$. In addition, using the measure Q that was introduced in Section 2.2, it was shown that

$$W = \frac{i^{0}¹0^{o}}{i^{0}_{Q}¹0^{o}}•$$

We now focus on determining $\bar{\ }^{¢}$ ¡ 0, which, according to the results that were presented in Section 2.2, is the unique positive solution of

$$\bar{\ }^{¢} = \_ \ 1 \ 1¹ \ \bar{\ }^{¢o}•$$

Inserting the expressions for $\_$ and $1¹U^o$, it is seen, after some elementary algebra, that the equation in the previous display can be alternatively written as

$$\bar{\ }^{¢} = k¹l^{o} ¸ \frac{\bar{\ }^{¢} ¸ k¹l^{o}}{i¹ \ \bar{\ }^{¢o} \ \ l} \ l-$$

which is solved for $\bar{\ }^{¢} = \backslash^{¢}$ (i.e., $\bar{\ }^{¢}$ is such that $i¹ \ \bar{\ }^{¢o} = 0$). Notice that this means that both $?¹D^o$ and $?¹D^o$ have the same exponential decay rate, namely $\backslash^{¢}$. We thus conclude that, as D ! 1, $?¹D^o•?¹D^o$ converges to a constant $\mathbb{W}•W = W^{¢}_{l}$ (which is necessarily at most 1).

So as to determine $W^{¢}_{l}$, we need to evaluate $\mathbb{W}$. To this end, we first notice that the Laplace exponent of $. ¹C^o$ is

$$i \ ¹U^o = A \ U \ \ \_ \ ¹1 \ \ 1 \ ¹U^{oo} = U \ \ k¹l^{o}¹1 \ \ 1 \ ¹U^{oo}•$$

Using the results of Section 2.2,

$$\mathbb{W} = \frac{1 ¸ k¹l^{o} \ ¹1 \ ^{o0}¹0^{o}}{1 ¸ k¹l^{o} \ ¹1 \ ^{o0}¹ \ \backslash^{¢o}}• \tag{10.4}$$

Using the definition of $1 \ ¹U^o$, it follows after a bit of standard calculus that

$$¹1 \ ^{o0}¹0^{o} = \frac{1}{k¹l^{o}} ¸ \frac{i^{0}¹0^{o}}{l}•$$

Along the same lines, again by recalling results from Section 2.2, and using that $i_{Q}¹U^o = i¹U \ \ \backslash^{¢o}$ and hence $i^{0}_{Q}¹0^{o} = i^{0}¹ \ \backslash^{¢o}$, we find that

$$¹1 \ ^{o0}¹ \ \backslash^{¢o} = \frac{1}{k¹l^{o}} ¸ \frac{\backslash^{¢} ¸ k¹l^{o}}{k¹l^{o}} \frac{i^{0}_{Q}¹0^{o}}{l}•$$

The final step is to insert the expressions that we found for $(1 \;)'(0)$ and $(1 \;)'(1 \; \backslash^\star)$ into (10.4). We thus find that

$$\bar{W} = W\frac{k(l)}{k(l) + \backslash^\star},$$

In summary, we have established the following result.

Proposition 10.2 *Assume* $\; 2 \quad$. *As* $D \to \infty$,

$$\frac{?(D)}{?(D)} \to W^\star_l := \frac{k(l)}{k(l) + \backslash^\star},$$

The above result shows that, indeed, $W^\star_l \uparrow 1$ as the inspection rate l grows large. In addition, using that $i(U) \; AU + \_ \to 0$ as $U \to \infty$ implies that $k(\backslash) \; (\backslash + \_), A \to 0$ as $\backslash \to \infty$, it can be verified that

$$\lim_{l \to \infty} l(1 \quad W^\star_l) = A\backslash^\star.$$

which means that, for l large, $W^\star_l$ behaves as $1 \quad A\backslash^\star, l$. This relation can be used to determine a 'rule of thumb' by which one can determine the minimally required inspection rate l such that the information loss due to Poisson inspection is below a given threshold.

10.3 Length of first excursion

In this section we consider the setting in which the insurance firm goes bankrupt if the surplus' process first excursion below 0 lasts longer than a given threshold. To this end, we study the distribution of the time spent at a negative surplus (up to an exponentially distributed epoch with parameter V, that is), before the surplus turns positive again, which we denote by $+_V(D)$.

B *Quantity of interest.* As before we denote by $g(D)$ the ruin time, i.e., the first epoch $-_D(C)$ attains a negative value (or: the first epoch the net cumulative claim process attains a value larger than D). In addition, we define $*\;(D)$ as the length of the interval after $g(D)$ at which the surplus level $-_D(C)$ uninterruptedly attains a negative value (or: the net cumulative claim process $.\,(C)$ uninterruptedly attains a value above D). We formally define $+_V(D)$ by the random variable

$$+_V(D) := \min\{*\;(D).)_V \quad g(D)\} 1\{g(D) \leq)_V\},$$

This object is of interest in settings in which bankruptcy occurs when the length of the first excursion of $.\,(C)$ above D exceeds some threshold. The objective of this section is to describe the distribution of $+_V(D)$. We refer to Figure 10.3 for an illustration.

B *Definitions, preliminaries.* A crucial role is played by the overshoot over level D, in the scenario that $.\,{}^1C^{\circ}$ exceeds D before the killing epoch $)_V$. We know from Section 5.4 how to compute (through its transform)

$$\mathrm{P}^1.\ {}^1g^1D^{\circ\circ} \quad D\ 2\ \mathrm{d}H\text{-}\ g^1D^{\circ}\ 6\ )_V{}^{\circ};$$

we refer to the corresponding density by ${}^1D\text{-}\ H\text{-}\ V^{\circ}$• This means that, applying the memoryless property of the exponential distribution,

$$\mathrm{E}\,4^{\ U+_V{}^1D^{\circ}} = \int_0^{1} {}^1D\text{-}\ H\text{-}\ V^{\circ}\ \mathrm{E}\,4^{\ U\ \min f\ f^1H^{\circ}\text{-}) _V g}\ \mathrm{d}H\text{-}$$

with f^1H°, as in the proof of Lemma 1.1 and Chapter 5, the time it takes for $.\,{}^1C^{\circ}$ to decrease by at least H. As we have seen in the proof of Lemma 1.1, for any $H\ ¡\ 0$,

$$\mathrm{E}\,4^{\ U f^1H^{\circ}} = 4^{\ k^1U^{\circ}H}\text{•}$$

In addition, the following (elementary) result is useful.

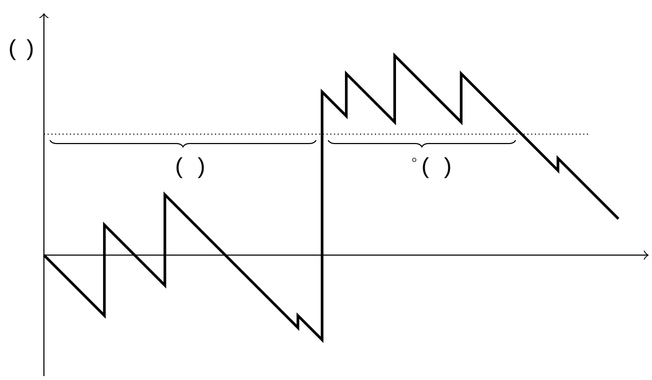

Fig. 10.3 Net cumulative claim process $.\,{}^1C^{\circ}$, and the quantities g^1D° and $*\ {}^1D^{\circ}$.

Lemma 10.3 *For any $U > 0$ and $V\ ¡\ 0$, and for any non-negative random variable $-$ that is independent of $)_V$,*

$$\mathrm{E}\,4^{\ U\ \min f\ -\text{-})_V g} = \frac{V}{U\ \text{¸}\ V}\ \text{¸}\ \frac{U}{U\ \text{¸}\ V}\,\mathrm{E}\,4^{\ {}^1U\text{¸}V^{\circ}\ -}\text{•}$$

Proof. There are various ways to prove this; we provide an analytical proof and a probabilistic one. In the analytical proof, the main idea is that we rewrite $\mathrm{E}\,4^{\ U\ \min f\ -\text{-})_V g}$ by conditioning on the value of $)_V$:

$$\int_0^{1} V4^{\ VC}\ \mathrm{E}\,4^{\ U\ \min f\ -\text{-}C g}\ \mathrm{d}C$$

$$= \int_0^1 V4^{VC} \int_0^C \mathrm{P}^1 - 2\, \mathrm{d}G^o 4^{UG} \mathrm{d}C \,¸ \int_0^1 V4^{VC} \int_C^1 \mathrm{P}^1 - 2\, \mathrm{d}G^o 4^{UC} \mathrm{d}C•$$

The stated follows by first swapping (in both terms) the order of the integrals, then evaluating the (easy) integrals over C, and finally interpreting the obtained expressions in terms of the LST of $-$.

The probabilistic proof relies on the identity $\mathrm{P}^1. \ddot{Y}\,)_V{}^o = \mathrm{E}4^{V.}$ for any non-negative random variable $.$ independent of $)_V$. Hence, for $)_U$ independent of $-$ and $)_V$,

$$\begin{aligned} \mathrm{E}4^{U \min f - -)_V g} &= \mathrm{P}^1 \min f - -)_V g\, \ddot{Y}\,)_U{}^o \\ &= \mathrm{P}^1)_V\, \ddot{Y}\,)_U{}^o \,¸\, \mathrm{P}^1)_U\, 6\,)_V{}^o\, \mathrm{P}^1 - \ddot{Y}\,)_U \mathrm{j})_U\, 6\,)_V{}^o \\ &= \mathrm{P}^1)_V\, \ddot{Y}\,)_U{}^o \,¸\, \mathrm{P}^1)_U\, 6\,)_V{}^o\, \mathrm{E}4^{1U¸V^o -} - \end{aligned}$$

where we have used that known fact that $)_U$ conditional on $)_U\ 6\ )_V$ is distributed as $)_{U¸V}•$

B *Derivation of transform of* $+_V{}^1D^o$. Upon combining the above (including the use of Lemma 10.3), we obtain

$$\begin{aligned} \mathrm{E}4^{U+_V{}^1D^o} &= \int_0^1 {}^1D\text{-}H\text{-}V^o \left(\frac{V}{U¸V} ¸ \frac{U}{U¸V} \mathrm{E}4^{1U¸V^o f^1H^o} \right) \mathrm{d}H \\ &= \frac{V}{U¸V}\, ?^1D\text{-})_V{}^o \,¸\, \frac{U}{U¸V} \int_0^1 {}^1D\text{-}H\text{-}V^o 4^{k^1U¸V^o H}\, \mathrm{d}H• \end{aligned}$$

Recalling the definition of ${}^1D\text{-}H\text{-}V^o$, we can interpret the integral in the previous display as the transform of the overshoot $.\ {}^1g^1D^{oo}\ D$ on the event $\mathrm{f}g^1D^o\ 6\ )_V \mathrm{g}$, so that we end up with the following result. We define

$$j^1D\text{-}U\text{-}V^o := \mathrm{E}\, 4^{k^1U¸V^o 1.\, ^1g^1D^{oo}\ D^o} 1\mathrm{f}g^1D^o\ 6\ )_V \mathrm{g}\ •$$

Proposition 10.3 *For any $U > 0$ and V ¡ 0,*

$$\mathrm{E}4^{U+_V{}^1D^o} = \frac{V}{U¸V}\, ?^1D\text{-})_V{}^o \,¸\, \frac{U}{U¸V}\, j^1D\text{-}U\text{-}V^o• \tag{10.5}$$

Observe that the expressions appearing in the right-hand side of (10.5) can be assessed; more precisely, the transform (with respect to D) of both objects in the right-hand side can be computed in closed form. Indeed,

$$\int_0^1 4^{ID}\, ?^1D\text{-})_V{}^o\, \mathrm{d}D = c^1 I\text{-}V^o = \frac{1}{i^1I^o\ V} \frac{i^1I^o}{I} \frac{V}{k^1V^o} - \tag{10.6}$$

as computed in Section 1.3, whereas the transform of $j^1D\text{-}U\text{-}V^o$ follows from

$$\hat{}^1 I\text{-}V\text{-}W^o := \int_0^1 4^{ID}\, \mathrm{E}^1 4^{W1.\, ^1g^1D^{oo}\ D^o} 1\mathrm{f}g^1D^o\ 6\ )_V \mathrm{g}^o\, \mathrm{d}D$$

$$= \frac{\bar\lambda}{\psi(q)-\beta}\left(\frac{b(\varphi(\beta))-b(\gamma)}{\gamma-\varphi(\beta)}-\frac{b(q)-b(\gamma)}{\gamma-q}\right), \tag{10.7}$$

using the analysis of Sections 5.3–5.4, leading to

$$\int_0^\infty e^{-qu}\,\pi(u,\alpha,\beta)\,\mathrm{d}u = \omega(q,\beta,\varphi(\alpha+\beta)). \tag{10.8}$$

By combining (10.5), (10.6), and (10.8), we conclude that

$$\int_0^\infty e^{-qu}\,\mathbb{E}\,e^{-\alpha\Upsilon_\beta(u)}\,\mathrm{d}u$$
$$=\frac{\beta}{\alpha+\beta}\frac{1}{\psi(q)-\beta}\left(\frac{\psi(q)}{q}-\frac{\beta}{\varphi(\beta)}\right)+\frac{\alpha}{\alpha+\beta}\,\omega(q,\beta,\varphi(\alpha+\beta)).$$

10.4 Total time with negative surplus

In this section we consider a setup in which the insurance firm goes bankrupt when the total time the surplus level is negative exceeds a certain threshold.

▷ *Definitions, preliminaries.* The metric that we consider here is the total time (until exponential killing) that the net cumulative claim process is larger than u:

$$\Upsilon_\beta(u) := \int_0^{T_\beta} 1\{X_u(t) < 0\}\,\mathrm{d}t = \int_0^{T_\beta} 1\{Y(t) > u\}\,\mathrm{d}t.$$

This quantity is of importance when bankruptcy depends on the time the insurance firm's surplus level has been below zero.

We analyze $\Upsilon_\beta(u)$ by evaluating its transform (with respect to u). Our argumentation follows the line of reasoning of the proof of Lemma 10.3. Observe that the following three disjoint events can occur: (i) $\{\sigma(u)+\tau(u)\leq T_\beta\}$, (ii) $\{\sigma(u)\leq T_\beta<\sigma(u)+\tau(u)\}$ and (iii) $\{T_\beta<\sigma(u)\}$.

Case (i) gives the contribution

$$\mathbb{E}\,e^{-\alpha\tau(u)}1\{\sigma(u)+\tau(u)\leq T_\beta\}\;\mathbb{E}\,e^{-\alpha\Upsilon_\beta(0)},$$

which is readily seen to equal $\pi(u,\alpha,\beta)\,\mathbb{E}\,e^{-\alpha\Upsilon_\beta(0)}$ (the reader is invited to verify this in Exercise 10.3).

Calling $\bar T_\beta$ the remaining part of T_β given that $T_\beta > \sigma(u)$, and using the memoryless property of the exponential distribution, the contribution due to Case (ii) can be seen to equal

$$\mathbb{E}\,e^{-\alpha\bar T_\beta}1\{\sigma(u)\leq T_\beta,\ \tau(u)>\bar T_\beta\}$$
$$=\mathbb{E}\,e^{-\alpha\bar T_\beta}1\{\sigma(u)\leq T_\beta\}-\mathbb{E}\,e^{-\alpha\bar T_\beta}1\{\sigma(u)\leq T_\beta,\ \tau(u)\leq\bar T_\beta\}$$

$$= \frac{V}{V ¸ U}\, ?¹D-)_V{}^{o} \quad \frac{V}{V ¸ U}\, j\,¹D-U-V^{o}•$$

Case (iii) finally contributes $P¹)_V\ Ÿ\ g¹D^{oo} = 1 \quad ?¹D-)_V{}^{o}$.

Adding the three contributions results in

$$E4^{\ U\,,\,v¹D^{o}} = j\,¹D-U-V^{o}\, E4^{\ U\,,\,v¹0^{o}}\ ¸\ 1$$
$$\frac{U}{U ¸ V}\, ?¹D-)_V{}^{o} \quad \frac{V}{U ¸ V}\, j\,¹D-U-V^{o}• \qquad (10.9)$$

Recalling that we can evaluate the transform (to D) of $?¹D-)_V{}^{o}$ and $j\,¹D-U-V^{o}$, we conclude that we are left with analyzing $E4^{\ U\,,\,v¹0^{o}}$.

B *Analysis of $,_v¹0^{o}$: a decomposition.* To compute $E4^{\ U\,,\,v¹0^{o}}$, we have to work with two auxiliary random sequences. Let $_8$ be the length of the 8-th uninterrupted period that $.\ ¹C^{o}$ is negative ('down'); likewise, we let $*_8$ be the length of the 8-th uninterrupted period that $.\ ¹C^{o}$ is non-negative ('up'). Observe that $¹\ \ _8-*_8{}^{o}{}_{82N}$ is a sequence of i.i.d. two-dimensional random vectors; let $¹\ \ -*^{o}$ denote the corresponding generic random vector. A pictorial illustration of these objects is provided in Figure 10.4. Then, exploiting the regenerative structure,

$$E4^{\ U\,,\,v¹0^{o}} = E\,4^{\ U*}1f\ \ ¸\ *\ 6\ )_V g\ \ E4^{\ U\,,\,v¹0^{o}}\ ¸$$
$$E\,4^{\ U¹)_V}\ {}^{o}1f\ \ 6\ )_V\ Ÿ\ \ ¸\ *g\ ¸\ P¹)_V\ Ÿ\ \ {}^{o}• \qquad (10.10)$$

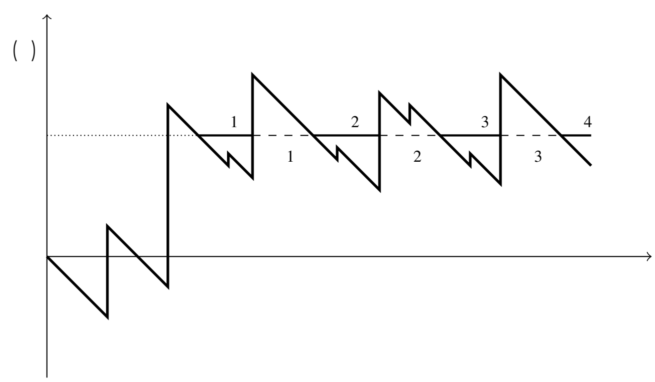

Fig. 10.4 Net cumulative claim process $.\ ¹C^{o}$, and the quantities $¹\ \ _8-*_8{}^{o}{}_{82N}$.

B *Evaluation of objects in decomposition.* We proceed by pointing out how the three unknown quantities can be evaluated.

We start with the analysis of the easiest object: $_1¹V^{o} := P¹)_V\ Ÿ\ \ {}^{o}$. We obviously have $P¹)_V\ Ÿ\ \ {}^{o} = 1 \quad P¹)_V > \ \ {}^{o}$, where $P¹)_V > \ \ {}^{o}$ can be rewritten as

$$\int_0^1 \_4^{1\_,V^oC} \int_0^{AC} P^1 \quad 2\, dE^o\, P^1)_V \,¡\, g^1AC \quad E^{oo} \,¸\, \int_{AC}^1 P^1 \quad 2\, dE^o \; dC$$

by conditioning on the first claim arrival time. Recalling that $P^1g^1D^o \; 6 \;)_V{}^o = ?^1D\text{-})_V{}^o$, performing the change of variable $B = AC$, and splitting the exponent, the expression in the previous display equals

$$\frac{\_}{A}\int_{1^0}^1 \int_{1^0}^{B} P^1 \quad 2\, dE^o\, 4^{1\_,V^o E\bullet A}\, ?^1B \quad E\text{-})_V{}^o\, 4^{1\_,V^o 1B \; E^o\bullet A}\, dB \,¸$$
$$\frac{\_}{A}\int_0^1 \int_B^1 P^1 \quad 2\, dE^o\, 4^{1\_,V^o B\bullet A}\, dB\bullet$$

Evaluating these integrals in the standard way, and recognizing the underlying convolution structure, we conclude that

$$_1{}^1V^o = \frac{V}{\_\,¸\,V} \,¸\, \frac{\_}{\_\,¸\,V}\, 1 \;\; \frac{\_\,¸\,V}{A} \quad \frac{\_}{A}\, 1 \;\; \frac{\_\,¸\,V}{A} \;\; c \;\; \frac{\_\,¸\,V}{A}\text{-}V \;\text{-}$$

with $c^1U\text{-}V^o$ as given in Section 1.3. After some straightforward calculus this turns out to simplify to

$$_1{}^1V^o = \frac{V}{Ak^1V^o}\bullet \qquad (10.11)$$

As an aside we remark that this final result could have been found directly as well, by applying Proposition 1.3 (plugging in $U = 0$).

We now focus on the evaluation of $\;_2{}^1U\text{-}V^o := E^14^{\;U^*}1f \quad ¸\; * \; 6 \;)_V g^o$. For conciseness, define by $._D$ the overshoot over level D, i.e., $.\,^1g^1D^{oo} \quad D\bullet$ Then $_2{}^1U\text{-}V^o$ can be rewritten, again by conditioning on the first claim arrival time, as

$$\int_0^1 \_4^{1\_,V^oC} \int_0^{AC} P^1 \quad 2\, dE^o \int_{AC\;\;E}^1 P^1 ._{AC\;\;E}\; 2\, dH\text{-}g^1AC \quad E^o \; 6 \;)_V{}^o$$
$$E^14^{\;U\,f\,^1H^o} 1f\,f\,^1H^o \; 6 \;)_V g^o \qquad !$$
$$¸\int_{AC}^1 P^1 \quad 2\, dE^o\, E^14^{\;U\,f\,^1E \;\; AC^o} 1f\,f\,^1E \quad AC^o \; 6 \;)_V g^o \;\; dC; \qquad (10.12)$$

here we distinguish between (i) the scenario in which after the first claim arrival (to happen before $)_V$) the net cumulative claim process is below level 0 (first term between the brackets), and (ii) the scenario in which at the first claim arrival the net cumulative claim process has exceeded level 0 (second term between the brackets); for an illustration, see Figure 10.5. The expression has been set up such that at the first claim arrival, at the end of , and at the end of $*$ the killing time $)_V$ has not expired.

Then observe that, for any $U > 0$ and $V \;¡\; 0$, and $H \;¡\; 0$,

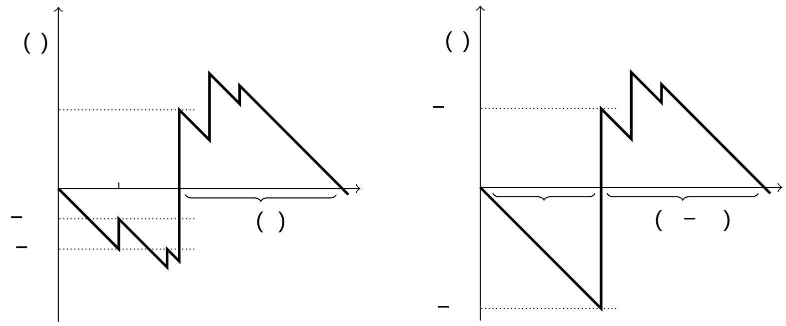

Fig. 10.5 Net cumulative claim process $. ^1C^o$ in a scenario that multiple claims are needed to exceed 0 (left panel), and in a scenario that one claim suffices (right panel). In the left panel the first jump (of size E Ÿ AC) happens at time C, eventually leading to an overshoot of H over level 0. In the right panel the first jump (of size $E > AC$) happens at time C, directly leading to an overshoot of $E \quad AC$ over level 0.

$$\mathrm{E}^1 4 \;^{U f ^1 H^o}\, 1 f\, f^1 H^o\, 6\,)_V g^o = \int_0^1 \int_0^C 4 \;^{UG} V 4 \;^{VC}\, \mathrm{P}^1 f^1 H^o\, 2\, \mathrm{d}G^o\, \mathrm{d}C$$
$$= \int_0^1 4 \;^{1U, V^o G}\, \mathrm{P}^1 f^1 H^o\, 2\, \mathrm{d}G^o$$
$$= \mathrm{E}\, 4 \;^{1U, V^o\, f^1 H^o} = 4 \;^{k^1 U, V^o H}\text{•}$$

Using this relation, and substituting B for AC, (10.12) can be written as the sum of two terms:

$$\frac{-}{A} \int_0^1 4 \;^{1-, V^o B \bullet A} \int_0^B \mathrm{P}^1 \quad 2\, \mathrm{d}E^o\, \mathrm{E}\, 4 \;^{k^1 U, V^o \cdot B' E}\, 1 f g^1 B \quad E^o\, 6\,)_V g^o\, \mathrm{d}B\text{-}$$

and

$$\frac{-}{A} \int_0^1 4 \;^{1-, V^o B \bullet A} \int_B^1 \mathrm{P}^1 \quad 2\, \mathrm{d}E^o\, 4 \;^{k^1 U, V^o\, ^1 E \quad B^o}\, \mathrm{d}B\text{•}$$

Recognizing the convolution structure, the first of these two terms can equivalently be written as

$$\frac{-}{A}\, 1 \quad \frac{-\,, V}{A} \quad \wedge \quad \frac{-\,, V}{A}\text{-}\, V\text{-}\, k^1 U \,,\, V^o \quad \text{•}$$

where the function $\wedge^1 \text{-} \text{-}\, ^o$ is defined in (10.7). The second term is easier to handle: after swapping the order of the integrals and some elementary calculus we readily obtain

$$- \frac{1^1 k^1 U \,,\, V^{oo} \quad 1^{11} - \,,\, V^o \bullet A^o}{- \,,\, V \quad A k^1 U \,,\, V^o}\text{•}$$

We conclude that

$$
{}_2{}^1U\text{-}\,V^{\text{o}} \;=\; \frac{-}{A}\,1\;\; \frac{-\,\text{¸}\,V}{A}\;\;\wedge\;\;\frac{-\,\text{¸}\,V}{A}\text{-}\,V\text{-}\,k^1U\;\text{¸}\;V^{\text{o}}\quad\text{¸}
$$
$$
-\;\frac{1^1k^1U\;\text{¸}\;V^{\text{oo}}\quad 1^{11}\_\;\text{¸}\;V^{\text{o}}\bullet A^{\text{o}}}{-\;\text{¸}\;V\quad Ak^1U\;\text{¸}\;V^{\text{o}}}\bullet
$$

Inserting the expression for $\wedge^1U\text{-}\,V\text{-}\,W^{\text{o}}$ that was given in (10.7), this expression greatly simplifies; we end up with

$$
{}_2{}^1U\text{-}\,V^{\text{o}} = \frac{-}{A}\;\frac{1^1k^1V^{\text{oo}}\quad 1^1k^1U\;\text{¸}\;V^{\text{oo}}}{k^1U\;\text{¸}\;V^{\text{o}}\quad k^1V^{\text{o}}}\bullet \tag{10.13}
$$

We conclude by computing ${}_3{}^1U\text{-}\,V^{\text{o}}$:= $\mathrm{E}^14^{\;U^1)_V}\;{}^{\text{o}}1\text{f}\quad 6\;)_V\;\ddot{Y}\quad\text{¸}\;*\text{g}^{\text{o}}$. Again by conditioning on the first claim arrival time,

$$
{}_3{}^1U\text{-}\,V^{\text{o}} = \int_0^{1}\!\!{}^{1}\; \_4^{\;1\_\text{¸}\,V^{\text{o}}C}\int_0^{1}{}^{AC}\;\mathrm{P}^1\quad 2\;\mathrm{d}E^{\text{o}}
$$
$$
{}^{1}\!\!\int_{AC\;E}^{1}\;\mathrm{P}^1\cdot_{\hat{AC}\;\;E}\;2\;\mathrm{d}H\text{-}\,g^1AC\quad E^{\text{o}}\;6\;)_V{}^{\text{o}}\,\mathrm{E}^14^{\;U)_V}1\text{f}\;f^1H^{\text{o}}\;\;i\;\;)_V\text{g}^{\text{o}}
$$
$$
!
$$
$$
\text{¸}\int_{AC}^{1}{}^{1}\;\mathrm{P}^1\quad 2\;\mathrm{d}E^{\text{o}}\,\mathrm{E}^14^{\;U)_V}1\text{f}\;f^1E\quad AC^{\text{o}}\;\;i\;\;)_V\text{g}^{\text{o}}\;\;\mathrm{d}C\bullet \tag{10.14}
$$

For any $U > 0$ and $V\;i\;0$, and $H\;i\;0$,

$$
\mathrm{E}^14^{\;U)_V}\,1\text{f}\;f^1H^{\text{o}}\;\;i\;\;)_V\text{g}^{\text{o}} = \int_0^{1}{}^{1}\;4^{\;UC}V4^{\;VC}\,\mathrm{P}^1f^1H^{\text{o}}\;\;i\;\;C^{\text{o}}\,\mathrm{d}C
$$
$$
= \frac{V}{U\;\text{¸}\;V}\,\mathrm{P}^1f^1H^{\text{o}}\;\;i\;\;)_{U\text{¸}V}{}^{\text{o}}
$$
$$
= \frac{V}{U\;\text{¸}\;V}\;\;1\quad 4^{\;k^1U\text{¸}V^{\text{o}}H}\;\bullet \tag{10.15}
$$

Using the same techniques as before, it can be verified that (10.14) can be written as

$$
{}_3{}^1U\text{-}\,V^{\text{o}} \;=\; \frac{-}{A}\,\frac{V}{U\;\text{¸}\;V}\,1\;\;\frac{-\,\text{¸}\,V}{A}\quad\wedge\;\;\frac{-\,\text{¸}\,V}{A}\text{-}\,V\text{-}\,0\quad\wedge\;\;\frac{-\,\text{¸}\,V}{A}\text{-}\,V\text{-}\,k^1U\;\text{¸}\;V^{\text{o}}\quad\text{¸}
$$
$$
\frac{\_V}{U\;\text{¸}\;V}\quad\frac{1\quad 1^{11}\_\;\text{¸}\;V^{\text{o}}\bullet A^{\text{o}}}{-\;\text{¸}\;V}\quad\frac{1^1k^1U\;\text{¸}\;V^{\text{oo}}\quad 1^{11}\_\;\text{¸}\;V^{\text{o}}\bullet A^{\text{o}}}{-\;\text{¸}\;V\quad Ak^1U\;\text{¸}\;V^{\text{o}}}\;\;\bullet
$$

After considerable calculus, we can show that this expression can be simplified to

$$
{}_3{}^1U\text{-}\,V^{\text{o}} = \frac{-}{A}\,\frac{V}{U\;\text{¸}\;V}\;\;\frac{1\quad 1^1k^1V^{\text{oo}}}{k^1V^{\text{o}}}\quad\frac{1^1k^1V^{\text{oo}}\quad 1^1k^1U\;\text{¸}\;V^{\text{oo}}}{k^1U\;\text{¸}\;V^{\text{o}}\quad k^1V^{\text{o}}}\;\;\bullet \tag{10.16}
$$

Remark 10.1 Formula (10.15) very well illustrates how careful one should be when performing 'sanity checks' corresponding to plugging in α and/or β equal to 0. Indeed, first sending α to 0 and then β gives $1 - e^{-\psi(0)x}$, whereas if one does it the other way around, then one gets 0. These outcomes are not necessarily the same. When the net-profit condition $\mathbb{E}\,Y(1) < 0$ is fulfilled (and also when $\mathbb{E}\,Y(1) = 0$), one has $\psi(0) = 0$, so that both limits match. When $\mathbb{E}\,Y(1) > 0$, however, one has that $\psi(0) > 0$, so that they do not match. When one thinks about this, this actually makes perfect sense: the first order should give $\mathbb{P}(\tau(x) = \infty)$, whereas for the second order realize that the quantity of interest is smaller than $\mathbb{E}\,e^{-\alpha\beta}$ which goes to 0 (for any $\alpha > 0$) when sending β to 0. ◇

▷ *Transform of* $\tau_\beta(0)$. Upon collecting the above results, and applying the decomposition (10.10), we have identified the transform of $\tau_\beta(0)$.

Theorem 10.3 *For any* $\alpha \geqslant 0$ *and* $\beta > 0$, $\mathbb{E}\,e^{-\alpha\tau_\beta(x)}$ *is given by* (10.9), *where* $\phi(x,\beta)$ *has transform* (10.6), $\kappa(x,\alpha,\beta)$ *has transform* (10.8), *and*

$$\mathbb{E}\,e^{-\alpha\tau_\beta(0)} = \frac{\xi_1(\beta) + \xi_3(\alpha,\beta)}{1 - \xi_2(\alpha,\beta)},$$

with $\xi_1(\beta)$ *given by* (10.11), $\xi_2(\alpha,\beta)$ *by* (10.13), *and* $\xi_3(\alpha,\beta)$ *by* (10.16).

10.5 Discussion and bibliographical notes

In Section 10.2 the situation is studied in which there is bankruptcy when the surplus level is below zero at a Poisson inspection epoch. The use of this bankruptcy concept was advocated in [5], acknowledging the fact that insurance firms sometimes can continue doing business even when they are technically ruined. In [8] the bankruptcy probability was determined for the Cramér-Lundberg setting with exponentially distributed claim sizes; in [9] this was extended to the case of generally distributed claim sizes. In both papers, the inspection rate was allowed to depend on the current surplus level. In [9], and also in [2], a related queueing (or inventory) model was also studied, where a server works even when there are no customers (or orders), building up storage that is removed at the Poisson inspection epochs.

Most of the material discussed in Section 10.2 was covered by [10]. That paper, however, considers the substantially broader framework of general Lévy processes. It for instance shows that the decomposition of Theorem 10.2 holds for *any* Lévy process, i.e., not just for processes $Y(t)$ that are of compound Poisson type. Various related topics are addressed in [10] as well, such as the tail asymptotics in the heavy-tailed case. In addition, an account of the relation with [7], in which related identities for Lévy processes at Poisson times are given, is provided. For other work on risk processes under a Poisson inspection mechanism, see e.g. [3, 4, 6]. The results of [10] have been further extended in [11].

The bankruptcy concept studied in Section 10.3 is based on the duration of the first excursion of the surplus process below 0. It mainly contains non-published material.

There is a direct relation with *Parisian ruin* [6, 17, 18], defined as the first time when an excursion of the surplus process below 0 is longer than some specified time (which is, for reasons of tractability, often assumed to be exponentially distributed); see also Exercise 10.5. Pioneering papers on Parisian ruin are [14, 15].

The bankruptcy concept of Section 10.4 is based on the total time that the surplus process spends below 0, in the literature sometimes referred to as *cumulative Parisian ruin* [16, 12], and this section also contains mainly new results. An alternative technique to identify expressions for the quantities that feature in Theorem 10.3, could be based on results for the G/M/1 queue that were derived in [1].

Exercises

10.1 ($\star$) Consider the bankruptcy model with Poisson inspection times, as introduced in Section 10.2. Define the two non-negative random quantities

$$Y^+(\beta,\omega) \equiv Y^+ := \bar{X}(T_{\beta+\omega}),\quad Y^-(\beta,\omega) \equiv Y^- := -\underline{X}(T_{\beta+\omega}).$$

In addition we write $\kappa(\alpha,\beta) := \mathbb{E}\, e^{-\alpha Y^+}$ (which can be evaluated by applying Theorem 1.1). Recall that Y^- is exponentially distributed with parameter $\nu := \psi(\beta+\omega)$, and that Y^- and Y^+ are, by virtue of the Wiener-Hopf decomposition (i.e., Proposition 1.2), independent.

In line with earlier definitions, we write

$$\bar{\pi}(\alpha,\beta) := \int_0^\infty e^{-\alpha u}\, p(u, T_\beta)\, \mathrm{d}u.$$

The objective of this exercise is to develop an alternative, more intuitive derivation of Theorem 10.1, based on 'first principles'.

(i) Argue that

$$p(u,T_\beta) = \frac{\omega}{\beta+\omega}\left(\mathbb{P}(Y^+ - Y^- < u) + \int_{-\infty}^{u} \mathbb{P}(Y^+ - Y^- \in \mathrm{d}v)\, p(u-v, T_\beta)\right).$$

(ii) Prove that

$$\int_0^\infty e^{-\alpha u}\, \mathbb{P}(Y^+ - Y^- < u)\, \mathrm{d}u = \frac{\alpha\,\kappa(\nu,\beta) - \nu\,\kappa(\alpha,\beta)}{\alpha(\nu-\alpha)} + \frac{1}{\alpha}.$$

(*Hint*: condition on the value of Y^- and swap the order of the two integrals.)

We proceed by splitting

$$\int_{u=0}^\infty e^{-\alpha u} \int_{v=-\infty}^{u} \mathbb{P}(Y^+ - Y^- \in \mathrm{d}v)\, p(u-v,T_\beta)\, \mathrm{d}u = \bar{\pi}_1(\alpha,\beta) + \bar{\pi}_2(\alpha,\beta),$$

where $\bar{c}_1{}^1U\text{-}\,V^{\text{o}}$ corresponds to the (triangular) integration area f0 6 E 6 Dg and $\bar{c}_2{}^1U\text{-}\,V^{\text{o}}$ to the (rectangular) integration area fD > 0-E 6 0g.
(iii) Show, again by conditioning on the value of $/$, that

$$\bar{c}_1{}^1U\text{-}\,V^{\text{o}} = \backslash\ \bar{c}^1U\text{-}\,V^{\text{o}}\frac{b^1U\text{-}\,V^{\text{o}} \quad b^1\backslash\text{-}\,V^{\text{o}}}{\backslash \quad U}\bullet$$

(iv) Argue why for E Ÿ 0

$$\mathsf{P}^1/_{\text{›}} \quad / \quad \ddot{Y}\, E\, \mathrm{j}\, / \quad ¡\ /_{\text{›}}{}^{\text{o}} = 4^{\backslash E};$$

in addition, recall that $\mathsf{P}^1/ \quad ¡\ /_{\text{›}}{}^{\text{o}} = b^1\backslash\text{-}\,V^{\text{o}}$.
(v) Use (iv) to show that

$$\bar{c}_2{}^1U\text{-}\,V^{\text{o}} = \backslash\ b^1\backslash\text{-}\,V^{\text{o}}\,\frac{\bar{c}^1U\text{-}\,V^{\text{o}} \quad \bar{c}^1\backslash\text{-}\,V^{\text{o}}}{\backslash \quad U}\text{-}$$

so that

$$\bar{c}_1{}^1U\text{-}\,V^{\text{o}}\ {}_{\text{¸}}\ \bar{c}_2{}^1U\text{-}\,V^{\text{o}} = \backslash\,\frac{\bar{c}^1U\text{-}\,V^{\text{o}}b^1U\text{-}\,V^{\text{o}} \quad \bar{c}^1\backslash\text{-}\,V^{\text{o}}b^1\backslash\text{-}\,V^{\text{o}}}{\backslash \quad U}\bullet$$

(vi) Compute $\bar{c}^1U\text{-}\,V^{\text{o}}$ in terms of the unknown constant $\bar{c}^1\backslash\text{-}\,V^{\text{o}}$. Determine this constant.
(vi) Use the expression for $\bar{c}^1U\text{-}\,V^{\text{o}}$ to verify the claim of Theorem 10.1.
(*Hint*: it is noted that $\bar{c}^1U\text{-}\,V^{\text{o}}$ and $\bar{r}_{\,I}{}^1U\text{-}\,V^{\text{o}}$ can be expressed in one another using the translation formula (1.1).)

10.2 Show that

$$\lim_{I\ !\ 1}\ I \quad \mathrm{E}\,.\bar{\ }^1)_V{}^{\text{o}} \quad \mathrm{E}\,._{V\text{-}\,I} \quad = \_\,\mathrm{E}\ \bullet$$

10.3 Provide the details of the derivation of Formula (10.9).

10.4 Verify Equations (10.11), (10.13), and (10.16).

10.5 ¹¢º In this exercise we focus on *Parisian ruin*. This means that the insurance company goes bankrupt as soon as the net cumulative claim process has an excursion above level D that is longer than an independently sampled exponentially distributed time $)_X$, for some parameter X ¡ 0. For an initial level D, we denote by $\mathsf{p}^1D\text{-}\,V\text{-}\,X^{\text{o}}$ the probability of Parisian ruin before $)_V$, where $)_V$ is sampled independently of anything else. The random variables * and are as defined in Section 10.4.
(i) Argue that the probability of bankruptcy in the first excursion, and in addition before the exponential clock $)_V$ rings, is

$$\mathsf{p}_0{}^1D\text{-}\,V\text{-}\,X^{\text{o}} := \mathsf{P}^1{+}_V{}^1D^{\text{o}} \ ¡\ )_X{}^{\text{o}} = 1 \quad \mathrm{E}\,4^{\ X+_V{}^1D^{\text{o}}}\bullet$$

(ii) Argue that the probability of bankruptcy during the second excursion, again also before the exponential clock $)_V$ rings, is

$$\mathsf{p}_1{}^1D\text{-}\,V\text{-}\,X^{\text{o}} := \bar{\mathsf{p}}_0{}^1D\text{-}\,V\text{-}\,X^{\text{o}}\,\mathsf{q}^1V\text{-}\,X^{\text{o}}\text{-}$$

where

$$\bar{p}_0{}^1D\text{-}\,V\text{-}\,X^o := P^1 *\ {}^1D^o\ 6\ )_X\text{-}\, g^1D^o\ ¸\ *\ {}^1D^o\ \ddot{Y}\ )_V{}^o\text{-}$$
$$q^1V\text{-}\,X^o := P^1\min f * \text{-}\,)_V \quad g\ i\ )_X\text{-} \quad 6\ )_V{}^o•$$

(iii) Show that

$$p^1D\text{-}\,V\text{-}\,X^o = p_0{}^1D\text{-}\,V\text{-}\,X^o\ ¸\ \bar{p}_0{}^1D\text{-}\,V\text{-}\,X^o \frac{q^1V\text{-}\,X^o}{1 \quad \bar{q}^1V\text{-}\,X^o}\text{-}$$

where

$$\bar{q}^1V\text{-}\,X^o := P^1 *\ \ddot{Y}\ )_X\text{-} \quad ¸\ *\ \ddot{Y}\ )_V{}^o•$$

(iv) Determine

$$\int_0^{1} 4^{\ UD}\, p^1D\text{-}\,V\text{-}\,X^o\, dD•$$

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Appendix A
Laplace transforms

In this appendix we discuss the Laplace transform (LT) and Laplace-Stieltjes transform (LST). They are important tools in insurance risk, queueing theory and several other domains within probability theory, and indeed they play an important role throughout this book. The appendix can be viewed as background material. We refer to it at several places in the manuscript.

A.1 Definitions

Let $f(\cdot)$ be a continuous, real-valued function in $x \geq 0$, such that $|f(x)| \leq a e^{bx}$, for $x \geq 0$, with a and b given constants.

Definition A.1 The *Laplace transform* (LT) of $f(\cdot)$ is

$$\phi(s) := \int_0^\infty e^{-sx} f(x)\,\mathrm{d}x, \tag{A.1}$$

with s complex-valued and $\mathrm{Re}\, s > b$.

Observe that $\phi(s)$ is a convex decreasing function for s real.

We are particularly interested in cases in which $f(\cdot)$ is the density of a non-negative random variable . For instance, if is exponentially distributed with parameter μ, then

$$\phi(s) = \int_0^\infty e^{-sx} \mu e^{-\mu x}\,\mathrm{d}x = \frac{\mu}{\mu + s}$$

for s such that $\mathrm{Re}\, s \geq 0$. As in the above definition, we can extend the region to $\mathrm{Re}\, s > -\mu$.

In the case that the function $f(\cdot)$ has discontinuities, one should replace the Riemann integral in (A.1) by a Riemann-Stieltjes integral, and accordingly speak of a Laplace-Stieltjes transform. This occurs for example when one considers the probability distribution of a random variable X with one or more atoms, or if one does not know whether the density of X exists everywhere.

Definition A.2 The *Laplace-Stieltjes transform* (LST) of $\mathbb{P}(X \leq x)$ is

$$\phi(s) := \int_0^\infty e^{-sx}\, \mathbb{P}(X \in \mathrm{d}x), \tag{A.2}$$

with $\operatorname{Re} s \geq 0$.

The use of an LST is required, for example, when X is the waiting time in a queueing model; its distribution typically has an atom at zero, representing the probability that a customer has zero waiting time. One sometimes writes

$$\phi(s) = \int_{0^-}^\infty e^{-sx}\, \mathbb{P}(X \in \mathrm{d}x),$$

to make explicit that this atom is included.

In probability theory, with a slight abuse of terminology, one often calls $\phi(s)$ as defined in (A.1) not just the LT of density $f(\cdot)$, but also the LT of the random variable X; analogously, $\phi(s)$ as defined in (A.2) is often referred to as the LST of the random variable X. If $\phi(s)$ is the LT (or LST) of a non-negative random variable X, then one can interpret $\phi(s)$ as an expectation, which offers many advantages, in terms of probabilistic manipulations and insights:

$$\phi(s) = \mathbb{E}\, e^{-sX}.$$

It is readily seen that the LT (or LST) of a non-negative non-defective random variable satisfies $\phi(0) = 1$ and, for all s with $\operatorname{Re} s > 0$,

$$|\phi(s)| \leq \int_0^\infty |e^{-sx}|\, \mathbb{P}(X \in \mathrm{d}x) \leq \int_0^\infty \mathbb{P}(X \in \mathrm{d}x) = 1.$$

Remark A.1 Generating functions are the discrete counterparts of Laplace transforms, and there is a very similar theory about them. Specifically, the probability generating function corresponding to the non-negative integer-valued random variable N is given by

$$\mathbb{E}\, z^N = \sum_{n=0}^\infty \mathbb{P}(N = n)\, z^n,$$

where $|z| \leq 1$. As we hardly use generating functions in this book, we do not treat them here.

A.2 Some convenient properties

In probability theory, LTs and LSTs have several convenient properties. In this section we discuss four such features.

▶ *Moments.* It is easy to obtain the moments of X from its LT or LST. In particular,

$$\mathrm{E}(X) = -\frac{\mathrm{d}}{\mathrm{d}s}W(s)\Big|_{s=0},$$

and more generally,

$$\mathrm{E}(X^n) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}W(s)\Big|_{s=0}.$$

For example, if X is exponentially distributed with parameter μ then repeated differentiation of $\mu/(\mu+s)$ yields

$$\mathrm{E}(X^n) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\frac{\mu}{\mu+s}\Big|_{s=0} = n!\frac{\mu}{(\mu+s)^{n+1}}\Big|_{s=0} = \frac{n!}{\mu^n}.$$

▶ *Sums of independent non-negative random variables.* The LT (LST) of the sum of a fixed number of independent non-negative random variables is the product of the individual LTs (LSTs, respectively). Indeed, if $X_1, \ldots, X_k$ are independent, then

$$\mathrm{E}\,e^{-s(X_1+\cdots+X_k)} = \mathrm{E}\,e^{-sX_1}\cdots e^{-sX_k} = \prod_{i=1}^{k}\mathrm{E}\,e^{-sX_i}. \tag{A.3}$$

The $\mathrm{E}\,e^{-sX}$ representation of $W(s)$ allows this fast proof.

More generally, if $g(\cdot)$ is the convolution of two non-negative functions $f_1(\cdot)$ and $f_2(\cdot)$ (like in the case of the density of a sum of two independent random variables), i.e.,

$$g(x) = \int_0^x f_1(y) f_2(x-y)\,\mathrm{d}y,$$

then, interchanging integrations, it easily follows that the LT of $g(\cdot)$ is the product of the LT's of $f_1(\cdot)$ and $f_2(\cdot)$:

$$\begin{aligned}
\int_{x=0}^{\infty} e^{-sx} g(x)\,\mathrm{d}x &= \int_{x=0}^{\infty} e^{-sx}\int_{y=0}^{x} f_1(y) f_2(x-y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{y=0}^{\infty} e^{-sy} f_1(y)\int_{x=y}^{\infty} e^{-s(x-y)} f_2(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{y=0}^{\infty} e^{-sy} f_1(y)\,\mathrm{d}y\int_{z=0}^{\infty} e^{-sz} f_2(z)\,\mathrm{d}z.
\end{aligned}$$

Example A.1 Suppose that $X_1,\dots,X_k$ are i.i.d., all being exponentially distributed with parameter μ. Then the LT of $Y^{(k)} := \sum_{i=1}^{k} X_i$ is given by

$$\mathrm{E}\,e^{-\alpha Y^{(k)}} = \prod_{i=1}^{k} \mathrm{E}\,e^{-\alpha X_i} = \left(\frac{\mu}{\mu+\alpha}\right)^k.$$

Let $E_k(\mu)$ correspond to an Erlang distribution with shape parameter k and scale parameter μ. Noticing that

$$\int_0^\infty e^{-\alpha x}\mu^k \frac{x^{k-1}}{(k-1)!}e^{-\mu x}\,\mathrm{d}x = \left(\frac{\mu}{\mu+\alpha}\right)^k \int_0^\infty e^{-(\mu+\alpha)x}(\mu+\alpha)^k \frac{x^{k-1}}{(k-1)!}\,\mathrm{d}x = \left(\frac{\mu}{\mu+\alpha}\right)^k,$$

where the last step follows from the fact that the integrand of the second integral is an $E_k(\mu+\alpha)$ density (or by repeatedly using integration by parts), we conclude that $Y^{(k)}$ has an $E_k(\mu)$ density. This last observation, or differentiation of the LT of $Y^{(k)}$ with respect to α, immediately yields that the first and second moment are given by

$$\mathrm{E}\,Y^{(k)} = \frac{k}{\mu}, \quad \mathrm{E}\,(Y^{(k)})^2 = \frac{k(k+1)}{\mu^2},$$

and hence $\mathrm{Var}\,Y^{(k)} = k/\mu^2$. ◊

B *Random sums of independent non-negative random variables*. Now assume that $Z = B_1 + \dots + B_N$, with $B_1, B_2, \dots$ i.i.d. with LST $\beta(\alpha)$, and N a non-negative integer-valued random variable that is independent of all B_i. Then

$$\begin{aligned}\mathrm{E}\,e^{-\alpha Z} &= \sum_{n=0}^{\infty} \mathrm{P}(N=n)\,\mathrm{E}\big[e^{-\alpha\sum_{i=1}^{N} B_i}\,\big|\,N=n\big] \\ &= \sum_{n=0}^{\infty} \mathrm{P}(N=n)\,(\beta(\alpha))^n = \mathrm{E}\big[(\beta(\alpha))^N\big]. \end{aligned} \tag{A.4}$$

Differentiation with respect to α thus yields that, with B denoting a generic random variable distributed as each of the random variables $B_1, B_2, \dots$,

$$\mathrm{E}\,Z = \mathrm{E}\,N\,\mathrm{E}\,B, \quad \mathrm{Var}\,Z = \mathrm{Var}\,N\,(\mathrm{E}\,B)^2 + \mathrm{E}\,N\,\mathrm{Var}\,B.$$

Example A.2 Once more take all X_i distributed exponentially with parameter μ; furthermore, take N (shifted) geometrically distributed with parameter $p \in (0,1)$: $\mathrm{P}(N=n) = (1-p)p^{n-1}$ for $n = 1,2,\dots$. Then, with $\beta(\alpha) := \mu/(\mu+\alpha)$,

$$\mathrm{E}\,e^{-\alpha Z} = \frac{(1-p)\beta(\alpha)}{1-p\beta(\alpha)} = \frac{(1-p)\mu}{(1-p)\mu+\alpha}.$$

We conclude that Z is distributed exponentially with parameter $\mu(1-p)$. It does make sense that Z is memoryless, as not only are the Y_i memoryless, but also N is memoryless:

$$\mathbb{P}(N > m + n \mid N > m) = \mathbb{P}(N > n)$$

for $m, n \in \{0, 1, 2, \dots\}$. ◇

Example A.3 In this example we consider the compound Poisson process. Define, for i.i.d. non-negative random variables $X_1, X_2, \dots$ with LT (or LST) $b(\alpha)$, the stochastic process

$$X(t) := \sum_{i=1}^{N(t)} X_i,$$

with $N(t)$ a Poisson process with rate λ. This process is called a compound Poisson process. The quantity $X(t)$ may, e.g., represent the cumulative claimed amount in $[0, t]$, in the Cramér-Lundberg model of insurance risk, or the total amount of offered work in $[0, t]$ in the M/G/1 queueing model. It follows from (A.4) that

$$\mathbb{E}\, e^{-\alpha X(t)} = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \big(b(\alpha)\big)^n = e^{-\lambda(1-b(\alpha))t},$$

for $t > 0$. ◇

▶ *Interpreting an LST as a probability*. At this stage we would like to point out an interesting and often useful interpretation of the LST $\phi(\alpha) = \mathbb{E}\, e^{-\alpha X}$. Denoting by T_α an exponentially distributed random variable with parameter α, which is assumed to be independent of everything else, one can interpret the LST as a probability:

$$\phi(\alpha) = \mathbb{P}(X \leq T_\alpha).$$

An immediate application of this observation provides us with an even faster proof of (A.3):

$$\mathbb{E}\, e^{-\alpha(X_1 + \dots + X_k)} = \mathbb{P}(X_1 + \dots + X_k \leq T_\alpha) = \prod_{i=1}^{k} \mathbb{P}(X_i \leq T_\alpha) = \prod_{i=1}^{k} \mathbb{E}\, e^{-\alpha X_i},$$

where the second equality follows from the memoryless property of the exponential distribution.

▶ *Solving differential, integral and integro-differential equations*. Another reason that LTs (and LSTs) are frequently used in many studies in insurance risk and queueing is that they are effective in solving certain classes of integral, linear differential and (convolution-type) integro-differential equations that naturally arise in these fields; to some extent this is indeed related to the convolution property mentioned above, dealing with convolution integrals that reflect sums of independent random variables. We will not provide a detailed account here, but in various chapters examples can be found. In this respect three useful properties are the following (cf. Exercise A.4). Let $\phi(\alpha)$ denote the LT of $f(x)$.

The LT of $x^n f(x)$:

$$\int_0^\infty e^{-\alpha x} x^n f(x)\,\mathrm{d}x = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\alpha^n}\phi(\alpha). \tag{A.5}$$

Also, the LT of $f'(x)$:

$$\int_0^\infty e^{-\alpha x} f'(x)\,\mathrm{d}x = \alpha\phi(\alpha) - f(0). \tag{A.6}$$

Finally, the LT of $\int_0^x f(y)\,\mathrm{d}y$:

$$\int_0^\infty e^{-\alpha x}\left(\int_0^x f(y)\,\mathrm{d}y\right)\mathrm{d}x = \frac{\phi(\alpha)}{\alpha}. \tag{A.7}$$

A.3 Some useful concepts and results

In this section we discuss some concepts and results that are related to Laplace transforms, and some specific forms of Laplace transforms, that we often come across in insurance risk and queueing, in particular in the context of this book.

▷ *The Laplace exponent.* Let us return to the above-mentioned compound Poisson process $Y(t)$, but now we subtract the deterministic drift rt (for some $r > 0$). In insurance risk, this rt component could represent the insurance firm's premium income in $[0, t]$, and in queueing theory the amount of work that the server can process in t time units. Then B_i typically represents a claim size (in the insurance context), or service requirement (in the queueing context), and throughout the book we denote the corresponding LST by $b(\alpha)$. Hence, with $X(t) := Y(t) - rt$ representing a compound Poisson process minus drift, we have

$$\mathbb{E}\, e^{-\alpha X(t)} = e^{(r\alpha - \lambda(1 - b(\alpha)))t}.$$

The function

$$\varphi(\alpha) := \log \mathbb{E}\, e^{-\alpha X(1)} = r\alpha - \lambda(1 - b(\alpha)), \tag{A.8}$$

is called the *Laplace exponent* of the $X(t)$ process. It plays a pivotal role in both the Cramér-Lundberg model and the M/G/1 queue.

At many places in the book we come across the expression $\varphi(\alpha) - q$, with $q \geq 0$ some parameter. Using the convexity of $\varphi(\alpha)$ for $\alpha > 0$, it is immediately clear that $\varphi(\alpha) - q$ has a unique zero for $\alpha \in [0, \infty)$.

▷ *Number of zeroes of expressions involving LTs.* There may be cases in which one would like to prove that some function of α, for example $\varphi(\alpha) - q$, has a particular number of zeroes in the whole, complex, right-half s-plane. Then Rouché's theorem, which we state now, is a useful tool.

Theorem A.1 (Rouché) *If $f_1(s)$ and $f_2(s)$ are analytic functions inside and on a closed contour C, and $|f_2(s)| < |f_1(s)|$ on C, then $f_1(s) + f_2(s)$ has exactly as many zeroes inside C as $f_1(s)$.*

Example A.4 We demonstrate the application of Rouché's theorem by proving that $\psi(s) - x$ has exactly one zero in the right-half s-plane, for every $x > 0$. To this end, take $f_1(s) := cs - \lambda - x$, $f_2(s) := \lambda\beta(s)$, and take C to be a semi-circle in the right-half plane with center at the origin and radius ρ, closed by the line segment $(-i\rho, i\rho)$ on the imaginary axis; and let $\rho \to \infty$. Then $f_1(s)$ and $f_2(s)$ are clearly analytic inside and on C. It is also clear that $|f_1(s)| > |f_2(s)|$ on the semi-circle. On the imaginary axis, $|f_2(s)| \leq \lambda$ while $|f_1(s)| \geq \lambda + x$. Hence the conditions of Rouché's theorem are fulfilled. Now observe that $f_1(s)$ has exactly one zero $s = (\lambda + x)/c$ in the right-half s-plane. Hence also $f_1(s) + f_2(s) = \psi(s) - x$ has exactly one zero in that half-plane. ^

B *Two frequently recurring LTs.* At many places in the book, we encounter the following LT expressions:

$$\frac{1 - \phi(s)}{s} \quad \text{and} \quad \frac{\phi(s) - \phi(x)}{x - s}.$$

Below we point out how such expressions typically arise. Consider the two-dimensional Laplace transform

$$\phi(s, x) := \int_0^\infty e^{-su} \int_0^\infty e^{-xt} \mathbb{P}(B > u + t)\, du\, dt.$$

Substituting $v = u + t$ and interchanging integrations quickly yields

$$\phi(s, x) = \frac{\phi(s) - \phi(x)}{x - s}. \tag{A.9}$$

In insurance risk, one encounters this two-dimensional transform for example in the following setting. Starting with initial capital u, and earning premiums at rate one, and assuming that the first claim arrives at time t, ruin occurs via this first claim if the claim size $B_1 > u + t$. Taking transforms gives the above formula (with, evidently, $\phi(\cdot)$ replaced by $\beta(\cdot)$, i.e., the LST of B_1).

Closely related to this is the following observation from renewal theory. Consider the i.i.d. non-negative random variables $X_1, X_2, \dots$, each of them distributed as the generic random variable X with LST $\phi(s)$. With

$$S_n := \sum_{i=1}^{n} X_i, \quad N(u) := \inf\{n \in \mathbb{N} : S_n > u\} \tag{A.10}$$

denoting the associated partial sum process and the first-entrance time of (x,∞), we define the *overshoot* and *undershoot* (with respect to level $x > 0$) by

$$\xi_+(x) := S_{\tau(x)} - x \quad \text{and} \quad \xi_-(x) := x - S_{\tau(x)-1},$$

respectively. Also, let the random variables ξ_+ and ξ_- be the limiting counterparts of $\xi_+(x)$ and $\xi_-(x)$, respectively, as $x \to \infty$. A well-known result from renewal theory, proven in Exercise A.5, is that, for $x, y > 0$,

$$\mathbb{P}(\xi_+ > x, \xi_- > y) = \int_{x+y}^{\infty} \frac{\mathbb{P}(B > z)}{\mathbb{E}B}\,\mathrm{d}z. \tag{A.11}$$

Hence, using the above integrations, the joint LST of ξ_+ and ξ_- is given by

$$\mathbb{E}\,e^{-\alpha\xi_+ - \beta\xi_-} = \frac{1}{\mathbb{E}B}\,\frac{\varphi(\alpha) - \varphi(\beta)}{\beta - \alpha}.$$

In particular,

$$\mathbb{E}\,e^{-\alpha\xi_+} = \mathbb{E}\,e^{-\alpha\xi_-} = \frac{1 - \varphi(\alpha)}{\alpha\,\mathbb{E}B}.$$

Taking $x = 0$ or $y = 0$ in (A.11), we observe that the density of ξ_+ and of ξ_- is given by $\mathbb{P}(B > w)/\mathbb{E}B$.

▷ *Asymptotic results*. Without providing details, we briefly discuss two classes of results.

- Tauberian theorems translate behavior of a LT near zero to the tail behavior of a random variable. Analogously, Heaviside results translate behavior of a LT at a pole in the left-half α-plane to the tail behavior of a (light-tailed) random variable (cf. the exp($-\mu t$) behavior corresponding to the $\alpha = -\mu$ pole of $\mu/(\mu + \alpha)$).
- Feller's (or Lévy's) convergence theorem allows one to obtain results regarding the limit $F(t)$ of a sequence of distribution functions $F_n(t)$ from the convergence of their LSTs $\varphi_n(\alpha)$ to an LST $\varphi(\alpha)$.

Theorem A.2 (Feller) *If, for all $\alpha > 0$, $\varphi_n(\alpha) \to \varphi(\alpha)$ as $n \to \infty$, then $\varphi(\alpha)$ is the LST of a possibly improper random variable with distribution function $F(\cdot)$, and $F_n(t) \to F(t)$ at every continuity point of $F(t)$.*

▷ *Transform inversion*. An important property of LTs and LSTs is that the original function is uniquely determined by the transform; in particular, the LT (or LST) of a non-negative random variable uniquely characterizes its distribution. While knowledge of the LT or LST therefore already provides valuable information, it is evident that ideally one would like to be able to invert the transform so as to obtain more explicit insight into the function or distribution one is interested in.

In some cases such explicit inversion is straightforward. Examples are the transforms of the types

$$\left(\frac{\mu}{\mu + \alpha}\right)^k, \quad \frac{p\,\mu_1}{\mu_1 + \alpha} + \frac{(1-p)\,\mu_2}{\mu_2 + \alpha};$$

recognize in the former transform the $E_2(\mu)$ distribution, and in the latter a mixture of two exponential distributions (with parameters μ_1 and μ_2, that is). More generally, transforms that are (sums of) rational functions of s, i.e., quotients of polynomials in s, can be reduced to known LT expressions by partial fraction expansion. Another useful insight is that, as we have seen above, the inverse of $(1-\phi(s))/s$ is $\mathbb{P}(X > x)$.

However, in most cases explicit inversion is challenging. Fortunately, in the last thirty years much progress has been made regarding the numerical inversion of LTs of non-negative random variables, which has led to easily-implementable, fast and accurate algorithms. Such numerical inversions are based on the numerical evaluation of an integral that features in a formal inversion formula which has been known for a long time. In particular, the inversion of $\phi(s)$ from (A.1) is given by

$$f(t) = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{c-iT}^{c+iT} e^{st}\phi(s)\,\mathrm{d}s, \tag{A.12}$$

where c is chosen such that all singularities of $\phi(s)$ lie (in the complexe plane) left of the integration line. In the case of the Laplace transform of a probability density, one can choose $c = 0$. By closing the integration line on its left side via a semi-circle with center at c and radius T, letting $T \to \infty$, one obtains a closed contour $\mathcal{C}$. A procedure for the explicit evaluation of this contour integral, that is being traversed counterclockwise, is to use *Cauchy's residue theorem*. If the integrand of (A.12) is analytic inside contour $\mathcal{C}$ except for a number of poles, then the contour integral equals the sum of the residues of these poles. Finally one needs to evaluate the integral along the semi-circle; it typically can be shown to be zero for $T \to \infty$ (cf. Exercise A.12).

A.4 Discussion and bibliographical notes

Classic texts on LTs are the books of Doetsch [3] and Widder [6]. Feller's Volume II [4] contains accessible discussions of the use of LTs in probability theory, and in particular of the use of Tauberian theorems. A short account, in terms of concrete recipes, of Tauberian and Heaviside results is given in [2, Chapter VIII]. Pioneering work regarding the numerical inversion of transforms of random variables is done by Abate and Whitt (see, e.g., [1]). More recent refinements of this inversion approach were made by Den Iseger [5].

Exercises

A.1 Suppose that, for $\mu_1, \mu_2 > 0$,

$$\phi(s) = \frac{\mu_1}{\mu_1 + s}\,\frac{\mu_2}{\mu_2 + s}.$$

(i) Invert this LT.
(ii) Show that the inverse is the density of a non-negative random variable.
(iii) Determine the variance of this random variable, both via the LT and via the inversion.

A.2 Suppose that, for $\mu_1, \mu_2 > 0$ and $p \in (0,1)$,

$$\gamma(\alpha) = \frac{\mu_1\mu_2 + (p\mu_1 + (1-p)\mu_2)\alpha}{(\mu_1+\alpha)(\mu_2+\alpha)}.$$

(i) Invert this LT.
(ii) Show that the inverse is the density of a non-negative random variable.
(iii) Determine the variance of this random variable, both via the LT and via the inversion.

A.3 Let N be Bin(m,p) distributed, and let $Y_1, Y_2, \ldots$ be i.i.d. exponentially distributed with parameter μ. Let $Z := \sum_{i=1}^{N} Y_i$.
(i) Determine the LST of Z.
(ii) Use this expression to obtain the mean and variance of Z.

A.4 Verify relations (A.5), (A.6), and (A.7).

A.5 Let $X_1, X_2, \ldots$ be a sequence of non-negative i.i.d. random variables, distributed as the generic random variable X, and let $\gamma(\alpha) := \mathbb{E}\, e^{-\alpha X}$ the corresponding LST. In addition, we denote by S_n and $\tau(u)$ the partial sum process and the first-entrance time of that partial sum process in (u,∞), respectively; see (A.10). We define the *overshoot* over level u by

$$R_+(u) := S_{\tau(u)} - u.$$

The objective of this exercise is to find the limiting distribution of $R_+(u)$ as $u \to \infty$. We consider the corresponding undershoots as well.
(i) Show that, for any $u > 0$,

$$\mathbb{E}\, e^{-\alpha R_+(u)} = \int_0^u \mathbb{P}(X \in \mathrm{d}y)\, \mathbb{E}\, e^{-\alpha R_+(u-y)} + \int_u^\infty \mathbb{P}(X \in \mathrm{d}y)\, e^{-\alpha(y-u)}.$$

(ii) Define

$$\sigma(\delta,\alpha) := \int_0^\infty e^{-\delta u}\, \mathbb{E}\, e^{-\alpha R_+(u)}\, \mathrm{d}u.$$

Prove that

$$\sigma(\delta,\alpha) = \gamma(\delta)\,\sigma(\delta,\alpha) + \frac{\gamma(\alpha)-\gamma(\delta)}{\delta-\alpha}.$$

(iii) Argue that, with R_+ the limit of $R_+(u)$ as $u \to \infty$,

$$\mathbb{E}\, e^{-\alpha R_+} = \lim_{\delta \downarrow 0} \delta\, \sigma(\delta,\alpha) = \frac{1}{\mathbb{E}X}\,\frac{1-\gamma(\alpha)}{\alpha}.$$

(iv) Conclude that, for $x > 0$,

$$P(X^+ \le x) = \int_0^x \frac{P(B > y)}{E(B)}\, dy.$$

(v) Define the *undershoot* by $X^-(t) := t - S_{N(t)-1}$. Show that, with X^- the limit of $X^-(t)$ as $t \to \infty$, for $x > 0$,

$$P(X^- \le x) = P(X^+ \le x).$$

(*Hint*: Use a similar approach as in proving the overshoot result.)
(vi) Show that

$$E e^{-sX^+ - tX^-} = \frac{1}{E(B)} \frac{\phi(s) - \phi(t)}{t - s}$$

and

$$P(X^+ \ge x_1, X^- \ge x_2) = 1 - \int_0^{x_1 + x_2} \frac{P(B > y)}{E(B)}\, dy.$$

(vii) Finally prove that

$$E e^{-s(X^+ + X^-)} = -\frac{\phi'(s)}{E(B)}$$

and

$$P(X^+ + X^- \le x) = \int_0^x \frac{y P(B \in dy)}{E(B)}.$$

A.6 Let B be exponentially distributed with parameter μ.
(i) Verify that

$$\phi(s, t) = \frac{\mu}{\mu + s} \frac{\mu}{\mu + t},$$

with $\phi(s, t)$ as given in (A.9).
(ii) Conclude that X^+ and X^- are in this case independent and both exponentially distributed with parameter μ.

A.7 Let B be $E_2(2\mu)$ distributed.
(i) Determine the LST of the residual lifetime X^+.
(ii) Use (i) to determine the density of X^+.

A.8 Consider the sequence of random variables B_n that are $E_n(n\mu)$ distrbuted. Use Feller's convergence theorem to show that this sequence converges to the degenerate (i.e., deterministic) random variable at $1/\mu$.
(*Hint*: The LST corresponding to B_n equals

$$\left(\frac{n\mu}{n\mu + s}\right)^n$$

for $s > 0$.)

A.9 Consider the integral equation, for $x \ge 0$,

$$A x\, E(x) = \lambda \int_0^x P(B \ge x - y)\, E(y)\, dy.$$

Here is a non-negative random variable with LST 1¹U°, and E¹ ° is the density of a non-negative random variable.
(i) Determine the LT of E¹ °.
(ii) Show that E¹ ° is a Gamma density if P¹ ¡ G° = 4 `G.
(iii) Let A = _ and P¹ ¡ G° = 4 `G. Verify that (ii) in this case yields the density of an exponentially distributed random variable with parameter `; also verify this result by substitution in the above integral equation.

A.10 Consider the following integral equation (due to Takács, for the steady-state workload + in the M/G/1 queue):

E¹G° = _ ∫₀^G P¹ ¡ G H° P¹+ 2 dH°- G ¡ 0•

Here E¹ ° is the density of the non-negative random variable + that has an atom at zero. The non-negative random variable has LST 1¹U°; the first moment of is less than 1•_ and the second moment of is finite.
(i) Determine the LST of +.
(ii) Use this LST to determine E +.
(*Hint*: while l'Hopital's rule works, it is easier to exploit the fact that

1¹U° = 1 UE ¸ (U²/2)E ² ¸ >¹U²°

as U # 0.)

A.11 Solve the following first-order linear differential equation by using LT s:

5⁰¹G° = 0 5 ¹G° ¸ 1-

with 5 ¹0° = 2.
(*Hint:* Handle the derivative by using partial integration.)

A.12 ¹¢° Invert W¹U° = ` •¹` ¸ U° by using Cauchy's residue theorem.

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Appendix B
Some queueing theory

In this appendix we discuss a few main results concerning queues with Poisson arrivals. We distinguish between the single-server queue of the type M/G/1 and the infinite-server queue of the type M/G/∞. The content of this appendix primarily serves as background, in the sense that knowledge of the presented results is strictly speaking not necessary when reading the book.

B.1 Single-server queue M/G/1

The M/G/1 queue is a single-server queue. Customers arrive at the service facility according to a Poisson process $A(t)$ with rate $\lambda > 0$, each of them requiring a particular amount of service. The service requirements form a sequence of i.i.d. random variables $B_1, B_2, \dots$ (also independent of the arrival process), distributed as a generic random variable B whose LST is given by $b(\alpha) = \mathbb{E}\,e^{-\alpha B}$. If an arriving customer finds the server busy, then it joins the queue. The size of the waiting room is assumed to be infinite. We consider the variant in which the server serves customers in order of arrival (usually referred to as *FCFS*, which stands for *First Come First Served*), working at speed $r > 0$ units of work per time unit. Throughout this appendix it is assumed that $\varrho := 1 - \lambda\,\mathbb{E}B / r > 0$. This condition, which states that the mean offered amount of work per time unit (i.e., $\lambda\,\mathbb{E}B$) is less than what can be served per time unit (i.e., r), is necessary and sufficient for the existence of the stationary distribution of the key performance measures. We discuss three of them: the stationary number of customers, the stationary workload, and the busy period. See Figure B.1 for an example of the evolution of the amount of work in the system $W(t)$ and (for the same arrival times and service requirements) the number of customers $Q(t)$.

▷ *Stationary number of customers.* The objective is to characterize the stationary number of customers. The idea is to embed a (discrete-time) Markov chain Q_n, viz. the number of customers in the system immediately after departures. It can be argued that the equilibrium distribution of N_n coincides with the equilibrium distribution of $N(t)$. Indeed, a simple up-and-downcrossing argument says that, in steady state, an arriving customer sees k customers in the system just as often as a departing customer leaves k customers behind, for any $k = 0, 1, \ldots$ (cf. the bottom panel of Figure B.1); and PASTA (Poisson Arrivals See Time Averages) implies that the stationary distribution of the number of customers seen by an arriving customer equals the stationary distribution of the number of customers at an arbitrary time.

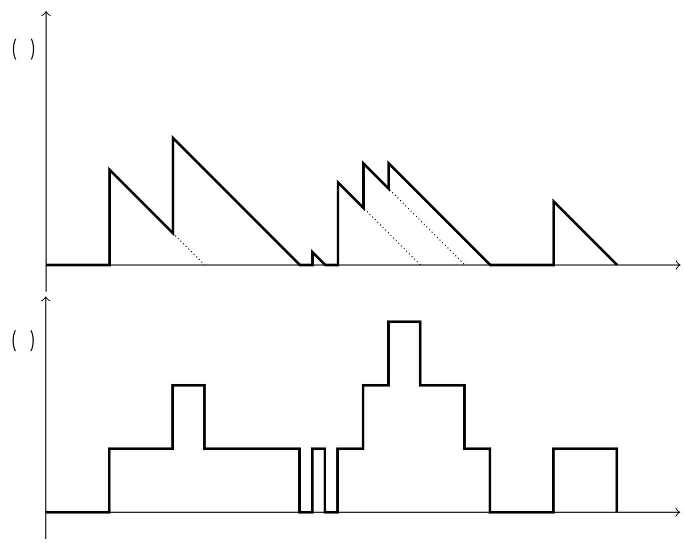

Fig. B.1 Workload $W(t)$ (top panel) in the M/G/1 queue, and corresponding number of customers $N(t)$ (bottom panel).

We let α_i denote the probability of $i = 0, 1, \ldots$ arrivals during a service time:

$$\alpha_i = \int_0^\infty P(A(t\lambda) = i)\, P(B \in dt) = \int_0^\infty e^{-t\lambda} \frac{(t\lambda)^i}{i!} P(B \in dt).$$

It can be checked that, with $p_{ij} := P(N_{n+1} = j \mid N_n = i)$, the transition probabilities of the embedded Markov chain are given by

$$p_{ij} = \begin{cases} \alpha_j & \text{if } i = 0, \\ \alpha_{j-i+1} & \text{if } i = 1, 2, \ldots \text{ and } j = i-1, i, i+1, \ldots, \\ 0 & \text{if } i = 2, 3, \ldots \text{ and } j = 0, 1, \ldots, i-2. \end{cases}$$

Let the random variable N represent the stationary number of customers in the system. Define $\pi_i := P(N = i)$, for $i = 0, 1, 2, \ldots$, i.e., the stationary distribution that we wish to determine, and

$$\Pi(z) := \sum_{i=0}^{\infty} \pi_i z^i$$

the corresponding probability generating function (see Remark A.1). In the sequel we also use the probability generating function

$$K(z) := \sum_{i=0}^{\infty} k_i z^i = \int_0^\infty e^{-\lambda t} \left(\sum_{i=0}^{\infty} \frac{(\lambda t)^i}{i!} z^i \right) P(B \in \mathrm{d}t)$$
$$= \int_0^\infty e^{-\lambda(1-z)t} P(B \in \mathrm{d}t) = \varphi(\lambda(1-z)).$$

With $P := (p_{ij})_{i,j=0}^{\infty}$, we have that $\boldsymbol{\pi} = \boldsymbol{\pi} P$, so that, for a given state $i = 0, 1, 2, \ldots$,

$$\pi_i = \sum_{j=0}^{\infty} \pi_j p_{ji}, \tag{B.1}$$

where the left-most expression can be interpreted as the stationary probability flux out of state i and the right-most expression as the stationary probability flux into state i. Hence

$$\Pi(z) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} \pi_j p_{ji} \right) z^i = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} \pi_{i-j+1} k_j + \pi_0 k_i \right) z^i$$
$$= \frac{1}{z} \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} \pi_{i-j+1} z^{i-j+1} k_j z^j + \pi_0 K(z)$$
$$= \frac{1}{z} \sum_{j=0}^{\infty} k_j z^j \left(\sum_{i=0}^{\infty} \pi_i z^i - \pi_0 \right) + \pi_0 K(z) = \frac{1}{z} K(z)(\Pi(z) - \pi_0) + \pi_0 K(z),$$

so that

$$\Pi(z) = \pi_0 \frac{K(z)(1-z)}{K(z) - z} = \pi_0 \frac{\varphi(\lambda(1-z))(1-z)}{\varphi(\lambda(1-z)) - z}.$$

The constant π_0 can be determined using the fact that $\Pi(1) = 1$: by applying l'Hopital's rule we obtain $\pi_0 = 1-\rho$. We thus end up with the following result, which is often referred to as the Pollaczek-Khinchine formula for the number of customers.

Theorem B.1 *In the* M/G/1 *queue, for any* $|z| \leqslant 1$,

$$\mathbb{E} z^{N} = (1-\rho) \frac{\varphi(\lambda(1-z))(1-z)}{\varphi(\lambda(1-z)) - z}.$$

▷ *Stationary workload.* We now concentrate on evaluating the stationary workload by applying a procedure that conceptually by and large coincides with the one used for the stationary number of customers. Clearly, we cannot work with identities of the type (B.1), focusing on the probability flux out of and into a single state, due to the uncountable state space of the workload. We therefore equate the probability flux out of and into the *set* $[x,\infty)$, for $x > 0$. More concretely, with $p(x)$ the density of the stationary workload W evaluated in the argument $x > 0$, we thus find the identity

$$r\,p(x) = \lambda \int_0^x \mathbb{P}(B > x-y)\,\mathbb{P}(W \in \mathrm{d}y). \tag{B.2}$$

Note that, with $p := \mathbb{P}(W=0)$,

$$\mathbb{P}(W \leqslant x) = p + \int_0^x \mathbb{P}(W \in \mathrm{d}y).$$

Then taking LT s on both sides of (B.2), and exploiting the fact that the LST (or LT) of a convolution is the product of the LST s (or LT s, respectively) of the individual terms, we readily obtain

$$r\,\mathbb{E}\,e^{-\alpha W} - p\,r = \lambda\,\mathbb{E}\,e^{-\alpha W}\,\frac{1-b(\alpha)}{\alpha}. \tag{B.3}$$

It follows directly from (B.3) that

$$\mathbb{E}\,e^{-\alpha W} = \frac{p\,r\,\alpha}{r\alpha - \lambda(1-b(\alpha))}. \tag{B.4}$$

In the denominator we recognize the Laplace exponent $\varphi(\alpha)$, as defined in (A.8). As inserting $\alpha = 0$ into the expression for $\mathbb{E}\,e^{-\alpha W}$ should yield 1, the constant p can be determined, and turns out to be $1-\varrho$ (in line with what we found in our analysis for the stationary number of customers). We thus obtain the following result, another version of the Pollaczek-Khinchine formula.

Theorem B.2 *In the* M/G/1 *queue, for any $\alpha > 0$,*

$$\mathbb{E}\,e^{-\alpha W} = \frac{\alpha\,\varphi'(0)}{\varphi(\alpha)}.$$

Remark B.1 The number of customers and the waiting time are the most studied performance measures in queueing theory. Workload is an ‘easier’ performance measure, because — unlike those other performance measures — it is insensitive to the service discipline, as long as the server works at a fixed speed whenever there is work. For the M/G/1 FCFS queue, there is an elegant argument to obtain the workload LST from the probability generating function of the queue length just after departures, derived in Theorem B.1. Firstly, observe that the N_n customers in the system just after the departure of the n-th customer are exactly those customers who have arrived during her time T_n in the system. Hence we have the following relation between the stationary queue length right after departures and stationary time in system: $\mathrm{E}\, I\,'' = \mathrm{E}\, 4\ {}^{-1}1\ \, I^{\circ})$. Secondly, $)$ is the sum of the waiting time and time in service of a customer. It now follows from Theorem B.1 that

$$\mathrm{E}\, 4\ {}^{UA\ ,}\ = 2\frac{U}{U\ \ {}^{1}\_\bullet A^{\circ}\ {}^{1}1\ \ 1^{1}U^{\circ\circ}}\bullet$$

Finally, PASTA implies that $A\ ,$ has the same distribution as $\&$; and indeed, the above formula agrees with (B.4). ^

B *Busy period.* In queueing systems, a busy period is defined as an uninterrupted time interval in which customers are served; busy periods alternate with idle periods, in which the system is empty (and which are, in the M/G/1 context, exponentially distributed with parameter _). We present an elegant argument for deriving the distribution of the length of such a busy period, f.

Consider the service time $_1\bullet A$ of the customer who starts a busy period by entering an empty system. Recall that $\#\,{}^1C^{\circ}$ is a Poisson process with rate _, representing the number of arrivals in an interval of length C. The idea is (i) to condition on the number of arrivals $\#\,{}^1\ {}_1\bullet A^{\circ}$ during this first service time, and (ii) to interrupt that service, first serving all the work arriving during this service time $_1\bullet A$ (i.e., to resume serving the first job when all other work that has arrived to the system has been served). Note that the duration of each interruption has the same distribution as the busy period itself (so that we can refer to the interruption as a 'sub busy period'), and that the number of interruptions is $\#\,{}^1\ {}_1\bullet A^{\circ}$. Thus, the busy period f can be written as the service time $_1\bullet A$, increased by $\#\,{}^1\ {}_1\bullet A^{\circ}$ 'sub busy periods':

$$f = \frac{1}{A}\ \text{¸}\ f_1\ \text{¸}\ \ \ \text{¸}\ f_{\#\,{}^1\ {}_1\bullet A^{\circ}};$$

see Figure B.2 for a pictorial illustration. Since the order in which customers are served is irrelevant for the length of the busy period (as long as the server serves when there is work in the system), and since the distribution of the length of any sub busy period is the same as the distribution of the actual busy period (and those are in addition independent), we can write by conditioning on $\#\,{}^1\ {}_1\bullet A^{\circ}$:

$$\begin{aligned}\mathrm{E}\, 4\ {}^{Uf} &= \int_{C=0}^{1} 4\ {}^{UC} \sum_{:=0}^{1} e^{-C}\frac{{}^1\_C^{\circ}{}^{:}}{:!} \prod_{8=1} \mathrm{E}\, 4\ {}^{Uf_8}\,\mathrm{P}^1\ {}_1\bullet A\ 2\ \mathrm{d}C^{\circ}\\ &= \int_{C=0}^{1} 4\ {}^{UC} \sum_{:=0}^{1} 4\ {}^{-C}\frac{{}^1\_C^{\circ}{}^{:}}{:!}\,{}^1\mathrm{E}\, 4\ {}^{Uf\circ :}\,\mathrm{P}^1\ {}_1\bullet A\ 2\ \mathrm{d}C^{\circ}\\ &= 1\ \frac{U}{A}\ \text{¸}\ \frac{-}{A}{}^1 1\ \ \mathrm{E}\, 4\ {}^{Uf\circ}\ \bullet\end{aligned}$$

It can be shown that, under the stability condition $2\ \ ¡\ 0$, there is a unique solution in $^1 0\text{-}\ 1¼$ to this fixed-point equation.

Theorem B.3 *In the* M/G/1 *queue, for any* $U > 0$, $\mathrm{E}\, 4\ {}^{Uf}$ *is the unique solution in* $^1 0\text{-}\ 1¼$ *of*

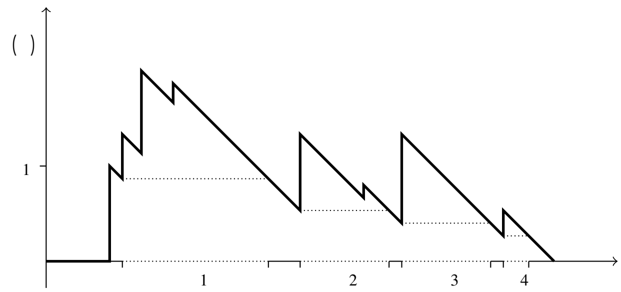

Fig. B.2 Busy period in the M/G/1 queue. In this case # ¹ $_1$•Aº = 4; the dotted segments correspond to the sub busy periods f_1-•••- f_4, all of them distributed as f. The solid segments on the horizontal axis add up to $_1$•A•

$$\mathrm{E}4^{\;Uf} = 1\;\frac{U}{A}\;¸\;\frac{—}{A}{}^{1}1\quad \mathrm{E}4^{\;Ufo}\;\;• \tag{B.5}$$

We can express E4 ^{Uf} more explicitly in terms of the inverse k¹\º of i¹Uº, as follows. Recall that, immediately from the definition of the Laplace exponent i¹Uº,

$$1^1U^o = \frac{i^1U^o\quad AU\;¸\;—}{—}•$$

Hence,

$$0 = 1\;\frac{U}{A}\;¸\;\frac{—}{A}{}^{1}1\quad \mathrm{E}4^{\;Ufo}\qquad \mathrm{E}4^{\;Uf} = \frac{1}{—}\;i\;\frac{U}{A}\;¸\;\frac{—}{A}{}^{1}1\quad \mathrm{E}4^{\;Ufo}\qquad \frac{U}{—}•$$

Multiplying this equation by _, we thus obtain

$$U = i\;\frac{U}{A}\;¸\;\frac{—}{A}{}^{1}1\quad \mathrm{E}4^{\;Ufo}\;\;•$$

Applying k¹º to both sides, we obtain after elementary calculus the following expression for the LST of the busy period, in terms of k¹º:

$$\mathrm{E}4^{\;Uf} = \frac{U\;¸\;—}{—}\quad \frac{A}{—}k^1U^o• \tag{B.6}$$

It is in addition noted that inversion of (B.5) turns out to be possible:

$$\mathrm{P}^1 f\;\ddot{Y}\;C^o = \int_0^{1}\sum_{:=1}^{C} e^{-D}\,\frac{{}^1\_D^{o:}\;1}{:!}\;\mathrm{P}\;\frac{1\;¸\qquad¸\quad :}{A}\;2\;\mathrm{d}D\;\;• \tag{B.7}$$

B.2 Infinite-server queue M/G/∞

Also in the M/G/∞ queue customers arrive according to a Poisson process and bring along i.i.d. service requirements that are distributed as a generic non-negative random variable B with LST $\beta(s)$. The distinguishing feature, however, is that all customers can be served in parallel: as there are infinitely many servers, the sojourn time of each customer coincides with her service requirement (where we for convenience assume a unit service rate). One could say that, in the literal sense, the M/G/∞ system is not a queue, as customers do not interfere: they do not have to wait before being served.

As it turns out, the stationary distribution of the number of customers can be computed using elementary properties of the Poisson process. As before, let $N(t)$ denote the number of customers present at time t, where we assume that the system is empty at time 0. Then the number of customer arrivals in $[0,t]$ is Poisson distributed with parameter λt. A well-known property of the Poisson process implies that each of them arrives at a uniformly distributed time in the interval $[0,t]$. As a consequence, the probability of an arbitrary arrived customer still being present at time t is, with U uniformly distributed on $[0,t]$,

$$p_t := \int_0^t \mathbb{P}(U \in \mathrm{d}x)\,\mathbb{P}(B > t - x) = \frac{1}{t}\int_0^t \mathbb{P}(B > t - x)\,\mathrm{d}x = \frac{1}{t}\int_0^t \mathbb{P}(B > x)\,\mathrm{d}x.$$

It thus follows, by virtue of the 'independent thinning property' of the Poisson process, that $N(t)$ has a Poisson distribution with parameter

$$\mu_t := \lambda t\, p_t = \lambda \int_0^t \mathbb{P}(B > x)\,\mathrm{d}x.$$

Note that

$$\lim_{t\to\infty} \mu_t = \lambda \int_0^\infty \mathbb{P}(B > x)\,\mathrm{d}x = \lambda\,\mathbb{E}B.$$

We thus have found the following result, which can be formally backed by the use of Theorem A.2.

Theorem B.4 *In the* M/G/∞ *queue, N is Poisson distributed with parameter $\lambda\,\mathbb{E}B$.*

B.3 Discussion and bibliographical notes

Comprehensive textbooks on queueing theory are e.g. [2, 4, 5, 6], but there are many others; see also the accessible lecture notes [1]. Specifically for the M/G/1 queue, see also the compact account [3]. A textbook that strongly focuses on computational aspects is [8].

Most results presented in this chapter are standard, and have been around for a long time. A textbook treatment of the infinite-server queue can be found in [7]. The inversion formula (B.7) appears in [4, p. 250].

Exercises

B.1 Derive the Pollaczek-Khinchine formula for the mean workload in the M/G/1 queue from the LST $\mathsf{E}\,4^{\ U\&}$:

$$\mathsf{E}\& = \frac{\mathsf{E}\ ^2}{2A2}\bullet$$

B.2 Determine the mean busy period $\mathsf{E}\,f$ in two different ways:
(i) By differentiation of (B.5) with respect to U.
(ii) By arguing probabilistically that

$$\mathsf{E}\,f = \frac{\mathsf{E}}{A} \ \text{¸} \ \frac{\mathsf{E}}{A}\,\mathsf{E}\,f\bullet$$

B.3 It can be concluded from the previous two exercises that the mean waiting time $\mathsf{E}\&\bullet A$ can be larger than $\mathsf{E}\,f$ for some service requirement distributions. Give a probabilistic explanation for this seeming paradox.

B.4 Let the service requirements be exponentially distributed with parameter \`.
(i) Rewrite (B.2) into

$$A@^1G^o = \int_0^{G} 4^{\ `^1G\ H^o} @^1H^o\,\mathrm{d}H \ \text{¸}\ 4^{\ `G} @- \tag{B.8}$$

with G ¡ 0, and solve this integral equation.
(*Hint*: introduce $@^1G^o := 4^{`G}@^1G^o$, multiply both sides of (B.8) by $4^{`G}$ and differentiate.)
(ii) Determine $@$.

B.5 Assume that the service requirements are exponentially distributed with parameter \`. Solve the functional equation (B.5) under this assumption.

B.6 Consider the M/G/1 queue, and assume that " ¹0º = 0.
(i) Prove that, for ¡ 0,

$$\log \mathsf{E}\, I_1^{\ "\ ^1C^o} I_2^{\ "\ ^1C\ ¸\ \ ^o} = \ A_{C\text{-}1}C^1 1 \ \ I_1{}^o \ ¸\ A_{C\text{-}2}C^1 1 \ \ I_1 I_2{}^o \ ¸\ A_{C\text{-}3}\ ^1 1 \ \ I_2{}^o \ \text{-}$$

where

$$A_{C\text{-}1} := \frac{1}{C}\int_0^{C} \mathsf{P}^1 D\ \ddot{Y}\ \ \ddot{Y}\ D\ ¸\ \ ^o\,\mathrm{d}D\text{-}$$

$$A_{C\text{-}2} := \frac{1}{C}\int_0^{C} \mathsf{P}^1\ \ ¡\ D\ ¸\ \ ^o\,\mathrm{d}D\text{-}$$

$$A_{C-3} := \frac{1}{\;} \int_0^1 P^1 \; ¡ \; D^o \, dD•$$

(ii) Show that, for ¡ 0, the limiting correlation coefficient between " ¹C º and " ¹C ¸ º satisfies

$$\lim_{C ! 1} \frac{\mathrm{Cov}^1 \; " \; ^1C^o \text{-} \; " \; ^1C \; ¸ \; ^{oo}}{\sqrt{\mathrm{Var} \; " \; ^1C^o \, \mathrm{Var} \; " \; ^1C \; ¸ \; ^o}} = \frac{1}{E} \int^1 P^1 \; ¡ \; D^o \, dD•$$

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